ON BOUNDED SEQUENCES AND SERIES CONVERGENCE IN BANACH ABSTRACT SPACE

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1. Introduction.

Theorem 3 gives two proofs of a sufficient condition for sequence convergence and series summability in normed or Banach abstract space. Such a condition amounts to a notion of sequence boundedness, called *d-boundedness* (deep boundedness), which is deeper than ordinary boundedness. In fact, first, any d-bounded sequence is bounded whereas the converse does not hold (Theorem 1). Second, at variance with ordinary boundedness, d-boundedness turns out to be more restrictive than convergence itself. In fact, d-boundedness implies convergence whereas, conversely, if and only if a particular condition is met then a convergent sequence is d-bounded (Theorem 4). In detail, Theorem 3, version (A), proves that a sufficient condition for sequence convergence and series summability in normed or Banach abstract space, is given by the following inequalities which define *sequence d-boundedness*

(1) $$\left\| s^{(n)} \right\| \le \mathrm{D}, \quad \left\| s_{(n+1)}^{(k)} \right\| \le \mathrm{D} \quad n,k = 1,2,\ldots, \left(\mathrm{D} = \text{const.} > 0\right)$$

where $s^{(n)}$ $n = 1,2,\ldots$ is a sequence in normed or Banach abstract space and the quantities $s_{(n+1)}^{(k)}$ are defined by such a sequence as follows

(2) $$s_{(n+1)}^{(k)} \equiv \frac{1}{k}\left(s^{(n+k)}(n+k) - s^{(n)} n \right) \quad n,k = 1,2,\ldots \quad \left(s_{(n+1)}^{(0)} \equiv 0 \right).$$

The theorems mentioned in the above are, *inter alia*, consequences of a simple representation result (Lemma 1) according to which any sequence $s^{(n)}$ $n = 1,2,\ldots$ defined in a linear space S can be represented as the sequence of the *CH-1 means* (i.e. Cesàro-Hölder first arithmetic means, see, e.g., **[Ces]**, **[Toe]**, **[Zyg]**, **[Wid**, pp. 311-2], **[Str**, pp. 474, 488])

(3) $$s^{(n)} = \frac{1}{n}\sum_{\nu=1}^{n} u^{(\nu)} \in S \qquad n = 1,2,\ldots$$

of a related and unique sequence $u^{(n)} \in S$, which we call the *underlying sequence* of the given sequence $s^{(n)} \in S$. Then, by (2)-(3), d-boundedness (1) actually amounts to the uniform boundedness of the following infinite-dimensional matrix

(4) $$\begin{array}{cccc} s^{(1)} = u^{(1)} & s^{(1)}_{(1+1)} = u^{(2)} & s^{(1)}_{(2+1)} = u^{(3)} & \cdots \\ s^{(2)} & s^{(2)}_{(1+1)} & s^{(2)}_{(2+1)} & \cdots \\ s^{(3)} & s^{(3)}_{(1+1)} & s^{(3)}_{(2+1)} & \cdots \\ \vdots & \vdots & \vdots & \end{array}$$

where (I) the first column is the given a sequence $s^{(n)} \in S$ or, by Lemma 1, sequence (3), (II) the first row is its underlying sequence $u^{(n)} \in S$, and (III) the remaining elements of (4) are (2) or, by (3), the moving averages

(5) $$s^{(k)}_{(n+1)} = \frac{1}{k}\sum_{\nu=n+1}^{n+k} u^{(\nu)} \in S \qquad n,k = 1,2,\ldots\ ;$$

then, setting $n = 1$ in (3) and $k = 1$ in (5) we get back the underlying sequence $u^{(n)} \in S$

(6) $$s^{(1)} = u^{(1)}, \quad s^{(1)}_{(n+1)} = u^{(n+1)} \qquad n = 1,2,\ldots\ ,$$

i.e., the first row of matrix (4); further, by (3), the elements of the first column are CH-1 means, and thus give rise to the corresponding Toeplitz matrix, **[Toe]**, **[Zyg]**, **[Wid]**. Moreover, it turns out (Theorem 2) that the d-boundedness (1) of a sequence $s^{(n)} \in S$, or the uniform boundedness of matrix (4), is equivalent to

(7) $$\left\| u^{(n)} \right\| \le \mathsf{D} \qquad n = 1,2,\ldots,$$

i.e., it is equivalent to the ordinary boundedness (7) of its underlying sequence $u^{(n)} \in S$. Thus, by Lemma 1 and Theorem 2, the class of the d-bounded sequences is not empty and, precisely, its members are all the sequences (3) of CH-1 means of all the bounded sequences. In fact, by Lemma 1 and Theorem 2, Theorem 3 (see version (A) above) has two further equivalent and meaningful versions (B)-(C). Indeed, on the one hand, (B) Theorem 3 proves that if the first row of matrix (4), i.e. any (possibly divergent) sequence $u^{(n)} \in S$, is bounded then such a sequence is CH-1 convergent (i.e. it converges in the sense of Cesàro-Hölder), namely the sequence (3) of the

CH-1 means of $u^{(n)} \in S$ (first column) converges in the ordinary sense. On the other hand, (C) Theorem 3 proves that the first column of matrix (4), i.e. any given sequence $s^{(n)} \in S$, converges in the ordinary sense if the first row, i.e. the underlying sequence $u^{(n)} \in S$, is bounded in the ordinary sense. Thus, by an approach actually designed for divergent sequences (or series), i.e. the Cesàro-Hölder method, Theorem 3 succeeds in proving a sufficient condition for the *ordinary* convergence of sequences (or summability of series) through the *ordinary* boundedness of the underlying sequences. Furthermore, as can be easily seen, version (B) agrees with a classical result of convergence and summability theory, see, e.g., **[Kno**, pp. 73-4**]**, **[Zyg**, p. 74**]**, **[Toe]**, **[Meg**, pp. 20, 372**]**, **[Woj**, p. 57**]**. However, a further interesting feature of the above convergence and summability results is that — Cauchy condition, Banach completeness and the Cesàro-Hölder means (3) excepted — their proofs do not involve any deep result or construct of the existing convergence and summability theory.

2. A type of sequence boundedness in Banach abstract space.

The introductory remarks of Section 1 motivate the following

DEFINITION 1. ("Deeply bounded" sequences in normed abstract space). *Let* $(S, \|\bullet\|)$ *be a normed space. Then, a sequence* $s^{(n)} \in S \quad n = 1,2,\ldots$ *is said to be* d-bounded (*deeply bounded*) *if* (1) *holds* ■

As a candidate for a sufficient condition of convergence, d-boundedness must exactly behave as proved by Theorem 1 below, that is: whenever a sequence $s^{(n)} \in S$ is d-bounded then it is bounded in the ordinary sense whereas the converse does not hold (a second proof of this fact will be given as a consequence of Theorems 3-4).

THEOREM 1. *Consider the abstract normed space* $(S, \|\bullet\|)$. *Then, for any sequence* $s^{(n)} \in S$ $n = 1,2,\ldots$

(i) *d-boundedness implies boundedness,*

(ii) *boundedness does not imply d-boundedness, and in particular we have*

(8)
$$\left\| s_{(n+1)}^{(k)} \right\| \leq \mathrm{D} + \frac{2\mathrm{D}\, n}{k} \qquad n, k = 1,2,\ldots.$$

First proof of Theorem 1. (i): *by* (1), *first inequality in particular*. **(ii)**: *by* (2) *and a property of the norm, we obtain*

(9) $$\left\| s_{(n+1)}^{(k)} \right\| \le \left(\left\| s^{(n+k)} \right\| (n+k) + \left\| s^{(n)} \right\| n \right) / k \, ;$$

thus, by the boundedness of the sequence $s^{(n)} \in S$, *i.e. the first inequality in* (1), *and simple algebra,* (9) *gives* (8) *that does not agree with the second inequality in* (1) ■

Theorems 2-4 rest on the simple representation Lemma 1 that we prove below. It states that any arbitrary sequence $s^{(n)}$ $n = 1,2,\dots$ defined in a linear space S can be represented as the sequence of the CH-1 means (Cesàro-Hölder first arithmetic means, see, e.g., **[Ces]**, **[Zyg]**, [**Wid**, pp. 311-2], [**Str**, pp. 474, 488]) of a related sequence $u^{(n)} \in S$ that we call the *underlying sequence* of the given sequence (of course is immaterial whether, in turn, the underlying sequence is seen as a sequence of partial sums of a series or not).

LEMMA 1. (Representation of sequences in linear abstract space by CH-1 means). *Consider the abstract linear space S and* $n = 1,2,\dots$. *Then, any sequence* $s^{(n)} \in S$ *can be represented in the form* (3), *i.e., as the sequence of the* CH-1 *means of some, and unique, sequence* $u^{(n)} \in S$ (*which is called the* underlying sequence *of the given sequence*). *Then, further,* (5)-(6) *hold*.

Proof. *Given a sequence* $s^{(n)} \in S$, *we get a unique sequence* $u^{(n)} \in S$ *by setting*

(10) $$u^{(1)} = s^{(1)}, \quad u^{(n)} = s^{(n)} n - s^{(n-1)} (n-1) \quad n = 2,3,\dots .$$

Furthermore, sequence (10) *gives back uniquely the given sequence* $s^{(n)}$ *represented in the form of the sequence of the* CH-1 *means of sequence* (10) *itself, i.e. in the form* (3). *Indeed, by obvious iterated inversions and substitutions,* (10) *gives uniquely*

$$s^{(1)} = u^{(1)}, \quad s^{(n)} = \left(s^{(n-1)} (n-1) + u^{(n)} \right) / n = \sum_{\nu=1}^{n} u^{(\nu)} / n \qquad n = 2,3,\dots$$

which is the given sequence $s^{(n)} \in S$ *in the form* (3). *Then, by* (2)-(3), (5) *holds, and, setting* $n = 1$ *in* (3) *and* $k = 1$ *in* (5), *we get* (6) ■

Theorem 2 below gives a necessary and sufficient condition for the d-boundedness of a sequence through the ordinary boundedness of its underlying sequence. In fact, Theorem 2 proves that the d-boundedness property (1) of a sequence $s^{(n)} \in S$ is equivalent to the ordinary boundedness (7) of its underlying sequence $u^{(n)} \in S$. Thus, by Lemma 1 and Theorem 2, the class of the d-bounded sequences is not empty and, precisely, its members are all the sequences (3) of CH-1 means of all the bounded sequences.

THEOREM 2. (Necessary and sufficient condition for sequence d-boundedness in normed abstract space). *Consider the normed abstract space* $(S,\|\bullet\|)$. *Let* $u^{(n)} \in S$ *be the underlying sequence of a sequence* $s^{(n)} \in S$ *and* $n = 1,2,\ldots$. *If and only if* $u^{(n)} \in S$ *satisfies* (7), *then* $s^{(n)} \in S$ *satisfies* (1).

Proof. *Necessity*. *By* (6) *of* Lemma 1, *if* (1) *holds then we get* (7). S*ufficiency*. *By* (3) *and* (5) *of* Lemma 1 *and the triangular property of the norm, if* (7) *holds then we get* (1); *indeed*

$$\left\| s^{(n)} \right\| = \left\| \sum_{n=1}^{n} u^{(n)} \right\| / n \le \sum_{n=1}^{n} \left\| u^{(n)} \right\| / n \le \mathsf{D}$$

$$\left\| s_{(n+1)}^{(k)} \right\| = \left\| \sum_{\nu=n+1}^{n+k} u^{(\nu)} \right\| / k \le \sum_{\nu=n+1}^{n+k} \left\| u^{(\nu)} \right\| / k \le \mathsf{D} \ \blacksquare$$

3. Convergence by sequence d-boundedness in Banach abstract space.

By two different proofs, Theorem 3 below shows that in a Banach space S an arbitrary sequence $s^{(n)} \in S$ is convergent to an element of S if

(a) by the first inequality in (1), the given sequence $s^{(n)} \in S$ is bounded, and moreover

(b) by the second inequality in (1), a much more deep boundedness condition is met, namely that the given sequence $s^{(n)} \in S$ is such that sequences (2), i.e. (5), be bounded by any bound D of the given sequence $s^{(n)} \in S$ (and vice-versa).

In particular Theorem 3 deals with series defined in the normed or Banach abstract space $(S,\|\bullet\|)$, that is

(11) $$\sum_{h=1}^{\infty} x^{(h)} \qquad x^{(h)} \in S \qquad h = 1,2,\ldots.$$

whose sequence of partial sums is

(12) $$s^{(n)} \equiv \sum_{h=1}^{n} x^{(h)} \in S \qquad n = 1,2,\ldots.$$

By its first proof, Theorem 3 shows that if (1) is satisfied by sequence (12) then series (11) satisfies the Cauchy necessary and sufficient condition for series summability, that is

(13) $$\left\| \sum_{h=n+1}^{n+k} x^{(h)} \right\| < \varepsilon \quad \forall n > \eta(\varepsilon) \quad \varepsilon \in (0,\infty) \quad k = 1,2,\ldots$$

or, which amounts to the same thing, sequence (12) satisfies the Cauchy necessary and sufficient condition for sequence convergence, that is

(14) $$\left\| s^{(n+k)} - s^{(n)} \right\| < \varepsilon \quad \forall n > \eta(\varepsilon) \quad \varepsilon \in (0,\infty) \quad k = 0,1,\ldots$$

for some function $\eta(\varepsilon)$, e.g. [**Die**, pp. 52-3, 95], [**Tré**, p. 52], [**Kno**, pp. 44, 67], [**Ban** pp. 9, 53]. As a by-product, Theorem 3 will also prove that (1) implies the necessary condition for series summability i.e., in particular, that

(15) $$\left\| x^{(1)} \right\| \leq \mathrm{D} \quad \left\| x^{(n)} \right\| \leq 2\mathrm{D}/(n-1) \qquad n = 2,3,\ldots \quad (\mathrm{D} = \text{const.} > 0).$$

By its second proof, which is based on the sufficiency part of Lemma 2A given in the Appendix, Theorem 3 shows that the Cauchy condition (14) holds for any d-bounded sequence $s^{(n)} \in S$. Thus, if such a sequence is the partial sum sequence (12), then the Cauchy condition (13) for series obviously follows. The second proof will be given in the Appendix at the end of the paper. We now state Theorem 3 and give its first proof.

THEOREM 3. (Sequences/series convergence/summability by d-boundedness in normed or Banach abstract space). *Consider the normed* (*Banach*) *abstract space* $(S,\|\bullet\|)$. *Then*

(i) *a series defined in* $(S,\|\bullet\|)$ *is summable* (*to an element of S*) *if the sequence of its partial sums is d-bounded,*

(ii) *a sequence defined in* $(S,\|\bullet\|)$ *is a Cauchy sequence* (*is convergent to an element of S*) *if it is d-bounded.*

First proof of Theorem 3. **(i)**: **(i.a)** *we prove that if* $(S,\|\bullet\|)$ *is a normed space and sequence* (12) *is such that* (1) *holds, then the Cauchy condition* (13) *for series summability holds as well*; *indeed, by* (1) *and* Theorem 2, (7) *holds; thus, by* (7), *the first inequality in* (1), *a property of the norm and identities* (1A) *of* Lemma 1A *in the* Appendix, *we get* (15), *that is*

$$\left\| x^{(1)} \right\| \leq \mathrm{D}, \left\| x^{(n)} \right\| \leq \left(\left\| u^{(n)} \right\| + \left\| s^{(n)} \right\| \right)/(n-1) \leq 2\mathrm{D}/(n-1), \; n = 2,3,\ldots \quad (\mathrm{D} = \text{const.} > 0);$$

further, by (15) *and the triangular property of the norm, we have*

(16) $$\left\| \sum_{h=n+1}^{n+k} x^{(h)} \right\| \leq \sum_{h=n+1}^{n+k} \left\| x^{(h)} \right\| \leq \sum_{h=n+1}^{n+k} 2\mathrm{D}/(h-1) < 2\mathrm{D}\,k/n, \quad n,k = 1,2,\ldots$$

that, as $n \to \infty$, *gives* (13) *with*

(17) $$h(e) = 2\mathrm{D}\,k\,e^{-1}, \quad e \in (0,\infty) \quad k = 1,2,\ldots;$$

indeed, by inversion of (17) *we obtain*

(18) $$\frac{2\mathrm{D}k}{\eta}=\varepsilon, \quad \eta=\eta(\varepsilon) \quad \varepsilon\in(0,\infty) \quad k=1,2,\ldots$$

from which we have

(19) $$\frac{2\mathrm{D}k}{n}<\varepsilon \quad \forall n>\eta=\eta(\varepsilon) \quad \varepsilon\in(0,\infty) \quad k=0,1,\ldots;$$

thus, by (18)-(19), *we have that* (16), *as* $n\to\infty$, *gives*

$$\left\|\sum\nolimits_{h=n+1}^{n+k}x^{(h)}\right\|<2\mathrm{D}k/n<2\mathrm{D}k/\eta=\varepsilon \quad \forall n>\eta(\varepsilon), \varepsilon\in(0,\infty), k=1,2,\ldots$$

which is (13) *with* (17). **(i.b)** *if* $(S,\|\bullet\|)$ *is a Banach space, by* (i.a) *and completeness, the assert is proved*. **(ii)**: **(ii.a)** *we prove that if* $(S,\|\bullet\|)$ *is a normed space, then for any sequence* $s^{(n)}\in S$ *such that* (1) *holds, the Cauchy condition* (14) *for sequence convergence holds as well; now, any given sequence* $s^{(n)}\in S$ *in a linear space S can be represented as the sequence of partial sums of some, and unique, series; indeed, given a sequence* $s^{(n)}\in S$ *we can set*

(20) $$s^{(1)}=x^{(1)}, \quad s^{(n)}-s^{(n-1)}=x^{(n)} \quad n=2,3,\ldots;$$

hence, by (20), *the given sequence* $s^{(n)}\in S$ *can be represented in the partial sum form*

(21) $$s^{(n)}=\sum\nolimits_{h=1}^{n}x^{(h)}\in S \qquad n=1,2,\ldots;$$

thus if the sequence $s^{(n)}\in S$ *is such that* (1) *holds then, by* (21) *and proposition* (i.a) *above, it satisfies* (13) *or, which amounts to the same thing, it satisfies* (14); **(ii.b)** *if* $(S,\|\bullet\|)$ *is a Banach space, by* (ii.a) *and completeness, the assert is proved* ■

Notice that, in the light of Lemma 1 and Theorem 2, Theorem 3 above (say, its version (A)) has two further equivalent and meaningful versions (B)-(C). Indeed, on the one hand, (B) Theorem 3 proves that if a (possibly divergent) sequence $u^{(n)}\in S$ is bounded, then such a sequence is CH-1 convergent (i.e. it converges in the sense of Cesàro-Hölder), namely the sequence (3) of the CH-1 means of $u^{(n)}\in S$ converges in the ordinary sense. On the other hand, (C) Theorem 3 proves that any given sequence $s^{(n)}\in S$ converges in the ordinary sense if its underlying sequence $u^{(n)}\in S$ is bounded in the ordinary sense. Thus, by an approach actually designed for divergent sequences (or series), i.e. the Cesàro-Hölder method, Theorem 3 succeeds in proving a sufficient condition for the *ordinary* convergence of sequences (or summability of series) through the *ordinary* boundedness of the underlying sequences. Moreover, as can be easily seen, version (B) agrees with a classical result of the convergence and summability theory, see, e.g., **[Kno**, pp. 73-4], **[Zyg**, p. 74], **[Toe]**, **[Meg**, pp. 20, 372], **[Woj**, p. 57]. However, a further interesting feature of the convergence and summability results given by Theorem 3 is that — Cauchy condition, Banach completeness and the Cesàro-Hölder means (3) excepted —its first proof above as well as its second proof in the Appendix do not involve any deep result or construct of the existing convergence and summability theory. The next result proves that, at variance with ordinary boundedness, d-boundedness is more restrictive than convergence itself. In fact, as proved by Theorem 4 below, a convergent sequence is d-bounded if and only if a particular condition is met.

THEOREM 4. *Consider the normed* (*Banach*) *abstract space* $(S, \|\bullet\|)$. *Then a Cauchy sequence* $s^{(n)} \in S$ *is d-bounded if and only if*

$$e(h) = \frac{k\,d}{h+k} \quad k = 1,2,\dots \quad (d = \text{const.} > 0) \tag{22}$$

where $\varepsilon(\eta)$ *is the inverse function of* $\eta(\varepsilon)$ *in the Cauchy condition* (14).

Proof. *Sufficiency. For any Cauchy sequence* $s^{(n)} \in S$ (14) *holds and, by* (22), *it gives*

$$\left\| s^{(n+k)} - s^{(n)} \right\| < e = \frac{k\,d}{h+k} \le \frac{k\,d}{n+k} \quad \forall n \le h(e), e \in (0,\infty),\ k = 1,2,\dots \tag{23}$$

$$h(e) = \left(de^{-1} - 1\right)k \to \infty \quad \text{as} \quad e \to 0\,; \tag{24}$$

thus (23)-(24) *give inequalities* (5A) (*see* Lemma 2A *in the* Appendix) *with* d=2D; *then, by the necessity part of* Lemma 2A, $s^{(n)} \in S$ *is d-bounded, that is*

$$\left\| s^{(n)} \right\| \le d/2, \quad \left\| s^{(k)}_{(n+1)} \right\| \le d/2 \quad n,k = 1,2,\dots, (d/2 = \text{const.} > 0). \tag{25}$$

Necessity. Conversely, if a Cauchy sequence $s^{(n)} \in S$ *is such that* (25) *holds then, by the second proof of* Theorem 3(ii) *in the* Appendix, *we get* (15A) *which is* (14) *with* (22), *or* (12A)-(13A), *and* d=2D ■

The second proof of Theorem 1 now follows as a consequence of Theorem 3 and the "only if" part of Theorem 4. Notice that the first proof of Theorem 3 does not make use of Theorem 1. As for the proof of the "only if" part of Theorem 4, it depends on the second proof of Theorem 3 which (as shown in the Appendix) depends on the sufficiency part of Lemma 2A whose proof does not make use of Theorem 1 either. For the reader's convenience, we repeat below, *mutatis mutandis*, the statement of Theorem 1.

THEOREM 1. *Consider the abstract normed space* $(S, \| \bullet \|)$. *Then, for any sequence* $s^{(n)} \in S$ $n = 1,2,\ldots$

(i) *d-boundedness implies boundedness,*

(ii) *boundedness does not imply d-boundedness.*

Second proof of Theorem 1. (i): *if a sequence* $s^{(n)} \in S$ *is d-bounded then, by* Theorem 3(ii), *is a Cauchy sequence and, therefore, is bounded.* **(ii)**: *consider a sequence* $s^{(n)} \in S$ *such that* (a) *it is a Cauchy sequence, and* (b) (22) *does not hold; then, by* (a), *the sequence* $s^{(n)} \in S$ *is bounded and, by* (b) *and the "only if" part of* Theorem 4, *it is not d-bounded* ■

4. Appendix.

Lemma 1A proves identities (1A)-(2A). Identities (1A) are referred to in the first proof of Theorem 3. Identities (2A) are referred to in the proof of Lemma 2A whose sufficiency part is referred to in the second proof of Theorem 3 at the end of this Appendix.

LEMMA 1A. (Identities for sequences/series in linear abstract space). *Consider series* (11) *in the linear abstract space S and the sequence of its partial sums* (12). *Then*

(1A) $$x^{(1)} = u^{(1)}, \quad x^{(n)} = \left(u^{(n)} - s^{(n)}\right) / (n-1) \quad n = 2,3,\ldots$$

where $u^{(n)} \in S$ *is the underlying sequence of sequence* (12); *further, for any sequence* $s^{(n)} \in S$

(2A) $$s^{(n+k)} - s^{(n)} = \frac{k}{n+k} \left(s^{(k)}_{(n+1)} - s^{(n)}\right) \quad n = 1,2,\ldots,\; k = 0,1,\ldots.$$

Proof of (1A). *Obviously, by* (12), *we have*

(3A) $$x^{(1)} = s^{(1)}, \quad x^{(n)} = s^{(n)} - s^{(n-1)} \quad n = 2,3,\ldots.$$

Further, by Lemma 1, (3) *and* (6) *hold for any sequence defined in S, then they hold for sequence* (12) *as well. Thus, by* (3), *the first equality in* (6) *and* (3A), *we have*

(4A) $$x^{(1)} = u^{(1)}, \quad x^{(n)} = \sum_{\nu=1}^{n} u^{(\nu)} / n - \sum_{\nu=1}^{n-1} u^{(\nu)} / (n-1) \quad n = 2,3,\ldots$$

where the first identity agrees with (1A), *and the second identity in* (4A) *gives*

$$x^{(n)} = \left(n u^{(n)} - \sum_{\nu=1}^{n} u^{(\nu)}\right) / n(n-1) \quad n = 2,3,\ldots$$

that is

$$x^{(n)} = u^{(n)} / (n-1) - \left(\sum_{\nu=1}^{n} u^{(\nu)} / n\right) / (n-1) \quad n = 2,3,\ldots$$

which, by (3), *is the second identity in* (1A). **Proof of (2A)**. *By* (3), *we have*

$$s^{(n+k)}-s^{(n)}=\sum_{\nu=1}^{n+k}u^{(\nu)}/(n+k)-\sum_{\nu=1}^{n}u^{(\nu)}/n$$

that is

$$s^{(n+k)}-s^{(n)}=\left(n\sum_{\nu=1}^{n}u^{(\nu)}+n\sum_{\nu=n+1}^{n+k}u^{(\nu)}-(n+k)\sum_{\nu=1}^{n}u^{(\nu)}\right)/n(n+k)$$

or

$$s^{(n+k)}-s^{(n)}=\left(n\,k\sum_{\nu=n+1}^{n+k}u^{(\nu)}/k-k\,n\sum_{\nu=1}^{n}u^{(\nu)}/n\right)/n(n+k),\quad (k=1,2,\dots)$$

which, by (3) *and* (5), *gives* (2A) ■

Lemma 2A proves inequalities (5A) which characterise d-bounded sequences. Further, the sufficiency part of Lemma 2A provides the basis for the second proof of Theorem 3 at the end of this Appendix.

LEMMA 2A. (Inequalities that characterise d-bounded sequences in normed abstract space). *Consider the normed abstract space* $(S,\|\bullet\|)$ *and let* $u^{(n)}\in S$ *be the underlying sequence of a sequence* $s^{(n)}\in S$ $n=1,2,\dots$. *If and only if sequence* $s^{(n)}\in S$ *satisfies* (1), *then*

(5A) $$\left\|s^{(n+k)}-s^{(n)}\right\|\le\frac{2\mathsf{D}k}{n+k}\quad n=1,2,\dots,\ k=0,1,\dots.$$

Proof. *Sufficiency. By* (1) *and a well-known property of the norm, we obtain*

(6A) $$\left\|s_{(n+1)}^{(k)}-s^{(n)}\right\|\le\left\|s_{(n+1)}^{(k)}\right\|+\left\|s^{(n)}\right\|\le 2\mathsf{D}\quad n=1,2,\dots,k=0,1,\dots;$$

thus, by (2A) *in norm, and* (6A), *inequalities* (5A) *hold. Necessity. Conversely, if* (5A) *holds, then, by* (2A) *in norm, we get* (6A); *further, by setting* $k=1$ *in* (6A) *and a well-known property of the norm, we have*

(7A) $$\left|\left\|s_{(n+1)}^{(1)}\right\|-\left\|s^{(n)}\right\|\right|\le\left\|s_{(n+1)}^{(1)}-s^{(n)}\right\|\le 2\mathsf{D}\quad n=1,2,\dots;$$

now we prove that if (5A) *and* (7A) *hold and* (1) *doesn't, then a contradiction arises; indeed, if* (1) *does not hold, then we can set*

(8A) $$\left\|s^{(n)}\right\|\le\mathsf{D},\quad\left\|s_{(n+1)}^{(1)}\right\|\ge\mathsf{D}+d\quad n=1,2,\dots,\quad d\in(0,\infty)$$

for any fixed positive d; thus, by the second formula in (8A), *and setting k*=1 *in* (8) *of* Theorem 1, *we get*

(9A) $$\mathsf{D}+d\le\left\|s_{(n+1)}^{(1)}\right\|\le(2n+1)\mathsf{D}\quad n=1,2,\dots,\quad d\in(0,\infty);$$

then, by (9A) *and the first formula in* (8A) *we have*

$$max\left|\left\|s^{(1)}_{(n+1)}\right\|-\left\|s^{(n)}\right\|\right| = max\left(\left|(2n+1)D - 0\right|, \left|(D+\delta)-D\right|\right)$$

that is

(10A) $$max\left|\left\|s^{(1)}_{(n+1)}\right\|-\left\|s^{(n)}\right\|\right| = max\left((2n+1)D,\ \delta\right)=(2n+1)D$$

(11A) $$\forall n > \left(\delta D^{-1}-1\right)/\,2, \quad \delta \in (0,\infty);$$

thus, by (10A)-(11A) *and* (7A), *we have the contradiction*

$$(2n+1)D \leq 2D, \quad \forall n > \left(\delta D^{-1}-1\right)/\,2, \quad \delta \in (0,\infty) \ \blacksquare$$

By its second proof, which is based on the sufficiency part of Lemma 2A, Theorem 3 shows that the Cauchy condition (14) holds for any d-bounded sequence $s^{(n)} \in S$ (Theorem 3(ii)); thus, if such a sequence is the partial sum sequence (12), then the Cauchy condition (13) for series obviously follows (Theorem 3(i)). For the reader's convenience, we repeat below the statement of Theorem 3.

THEOREM 3. (Sequences/series convergence/summability by d-boundedness in normed or Banach abstract space). *Consider the normed* (*Banach*) *abstract space* $(S,\|\bullet\|)$. *Then*

(i) *a series defined in* $(S,\|\bullet\|)$ *is summable* (*to an element of S*) *if the sequence of its partial sums is d-bounded,*

(ii) *a sequence defined in* $(S,\|\bullet\|)$ *is a Cauchy sequence* (*is convergent to an element of S*) *if it is d-bounded.*

Second proof of Theorem 3. **(ii)**: **(ii.a)** *we prove that if* $(S,\|\bullet\|)$ *is a normed space, then for any sequence* $s^{(n)} \in S$ *such that* (1) *holds the Cauchy condition* (14) *for sequence convergence holds as well; now, by* (1) *and the sufficiency part of* Lemma 2A, (5A) *holds; thus* (5A), *as* $n \rightarrow \infty$, *gives* (14) *with*

(12A) $$\eta(\varepsilon) = \left(2D\,\varepsilon^{-1}-1\right)k, \quad \varepsilon \in (0,\infty) \quad k = 0,1,\ldots;$$

indeed, by inversion, (12A) *gives*

(13A) $$\frac{2Dk}{\eta + k} = \varepsilon, \quad \eta = \eta(\varepsilon) \quad \varepsilon \in (0,\infty) \quad k = 1,2,\ldots$$

and, therefore, also gives

(14A) $$\frac{2Dk}{n+k} < \varepsilon \quad \forall n > \eta = \eta(\varepsilon) \quad \varepsilon \in (0,\infty) \quad k = 0,1,\dots;$$

thus, by (13A)-(14A), *we have that* (5A), *as* $n \to \infty$, *gives*

(15A) $$\left\| s^{(n+k)} - s^{(n)} \right\| \le \frac{2Dk}{n+k} < \frac{2Dk}{h+k} = e \quad \forall n > h(e) \quad k = 0,1,\dots$$

which is (14) *with* (12A). **(ii.b)**, *if* $(S, \|\bullet\|)$ *is a Banach space, by* (ii.a) *and completeness, the assert is proved.* **(i)**: **(i.a)** *let* $(S, \|\bullet\|)$ *be a normed space and* $s^{(n)} \in S$ *the sequence* (12) *of partial sums of a series such that* (1) *holds; therefore, by proposition* (ii.a), *the Cauchy condition* (14) *for sequences is satisfied by the sequence* $s^{(n)} \in S$ *; furthermore, since* $s^{(n)} \in S$ *is a sequence of partial sums then, by* (12), (14) *implies the Cauchy condition* (13) *for series;* **(i.b)** *if* $(S, \|\bullet\|)$ *is a Banach space, by* (i.a) *and completeness, the assert is proved* ■

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