MATRIX-VARIATE GROWTH-DECAY MODELS

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1. INTRODUCTION

If the input variable x_1 and the output variable x_2 are independently gamma distributed then it is known that the residual effect $y = x_1 - x_2$ has a density which can be expressed in terms of Whittaker functions in the scalar case. A particular case is a Laplace density. When x_j has the density

$$f_j(x_j) = \frac{x_j^{\alpha_j - 1} \mathrm{e}^{-x_j/\beta_j}}{\beta_j^{\alpha_j}\Gamma(\alpha_j)}, \ x_j > 0, \ \mathrm{Re}(\alpha_j) > 0, \ \mathrm{Re}(\beta_j) > 0, \ j = 1, 2 \tag{1.1}$$

and $f_j(x_j) = 0$ elsewhere, where $\mathrm{Re}(\cdot)$ denotes the real part of $(\cdot)$, the density of $y = x_1 - x_2$, denoted by $f(y)$, is given by

$$f(y) = \begin{cases} c_1 \ y^{\frac{\alpha_1+\alpha_2}{2}-1} \mathrm{e}^{-\frac{y}{2}\left(\frac{1}{\beta_1}-\frac{1}{\beta_2}\right)} \\ \times \ W_{\frac{\alpha_1-\alpha_2}{2}, \frac{1-\alpha_1-\alpha_2}{2}}(\beta_0 y), \ y > 0 \\ c_2 \ (-y)^{\frac{\alpha_1+\alpha_2}{2}-1} \mathrm{e}^{\frac{y}{2}\left(\frac{1}{\beta_2}-\frac{1}{\beta_1}\right)} \\ \times \ W_{\frac{\alpha_2-\alpha_1}{2}, \frac{1-\alpha_1-\alpha_2}{2}}(-\beta_0 y), \ y < 0 \end{cases} \tag{1.2}$$

where

$$\beta_0 = \frac{1}{\beta_1} + \frac{1}{\beta_2},$$
$$c_1^{-1} = \Gamma(\alpha_1)\beta_1^{(\alpha_1-\alpha_2)/2}\beta_2^{(\alpha_2-\alpha_1)/2}(\beta_1+\beta_2)^{(\alpha_1+\alpha_2)/2},$$
$$c_2^{-1} = \Gamma(\alpha_2)\beta_1^{(\alpha_1-\alpha_2)/2}\beta_2^{(\alpha_2-\alpha_1)/2}(\beta_1+\beta_2)^{(\alpha_1+\alpha_2)/2}$$

and $W_{\cdot,\cdot}(\cdot)$ is a Whittaker function, see also Mathai (1993a).

Independent sums of such residual effects are shown to be connected to the distributions of quadratic and bilinear forms in Gaussian random variables, see Mathai and Provost (1992), Mathai, Provost and Hayakawa (1995) and Rao (1973). Laplacianness of quadratic and bilinear forms or the conditions under which a quadratic or bilinear form is distributed as a Laplace variable or as a gamma difference is also investigated. Let S be a $p \times p$ non-singular central Wishart matrix and let it be partitioned as

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{1.3}$$

where S_{11} is $r \times r$, $r < p$ then it is shown that the distribution of S_{12} is associated with that of certain bilinear forms. Note that S_{12} can also be looked upon as a certain covariance structure when S is a corrected sample sum of products matrix. A large number of results in this category may be seen from Mathai (1993b) and Mathai, Provost and Hayakawa (1995).

Solar neutrinos are captured on earth and the data for over twenty years are available. The summarized data, by taking a five point moving average to smooth out local disturbances, show a peculiar pattern of a 9-year cyclic behavior where within each cycle the pattern is a slow rising and rapidly decreasing curve with one real peak and several small humps in between. This type of pattern is also seen in many other problems such as the production of melatonin in human body. In Haubold and Mathai (1995) it is shown that this type of curve could be generated by combining three residual variables where the input and output variables are of gamma type, and some heuristic interpretations of these variables are also given there.

One matrix-variate analogue of gamma type input and gamma type output gives the residual variable $Y = X_1 - X_2$ where X_1 and X_2 are independently distributed $p \times p$ matrix-variate real gamma random variables with the densities

$$g_j(X_j) = \frac{|B_j|^{\alpha_j}}{\Gamma_p(\alpha_j)} |X_j|^{\alpha_j - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(B_j X_j)}, \tag{1.4}$$

for $X_j = X_j' > 0$, $B_j = B_j' > 0$, $\mathrm{Re}(\alpha_j) > \frac{p-1}{2}$, and $g_j(X_j) = 0$, $j = 1, 2$ elsewhere, where for example,

$$\Gamma_p(\alpha) = \pi^{p(p-1)/4} \Gamma(\alpha) \Gamma\left(\alpha - \frac{1}{2}\right) ... \Gamma\left(\alpha - \frac{p-1}{2}\right), \ \mathrm{Re}(\alpha) > \frac{p-1}{2} \tag{1.5}$$

is the matrix-variate gamma function in the real case, with the integral representation

$$\Gamma_p(\alpha) = \int_{X = X' > 0} |X|^{\alpha - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(X)} \mathrm{d}X. \tag{1.6}$$

When $B_j = \frac{1}{2} A_j$ for some $A_j = A_j' > 0$ one has the non-singular central Wishart density from (1.4). Then if X_1 and X_2 are two independent central Wishart matrices then the residual variable is

$$Y = X_1 - X_2.$$

One can look at element-wise differences in $X_1 - X_2$ and consider the configuration of residual variables or one can consider the difference in terms of the definiteness of the $p \times p$ matrices X_1 and X_2. Since X_1 and X_2 are real symmetric positive definite matrices in (1.4), Y could be positive definite or negative definite or indefinite or semidefinite. A special situation will be when X_1 and X_2 are oriented matrices such that Y is either nonnegative definite or $-Y$ is nonnegative definite. In this case one can proceed in the following way to compute the density of Y. The joint density of X_1 and X_2, denoted by $f(X_1, X_2)$, is the product of $f_1(X_1)$ and $f_2(X_2)$ due to statistical independence. That is,

$$f(X_1, X_2) \mathrm{d}X_1 \mathrm{d}X_2 = \delta |X_1|^{\alpha_1 - \frac{p+1}{2}} |X_2|^{\alpha_2 - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(B_1 X_1 + B_2 X_2)} \mathrm{d}X_1 \mathrm{d}X_2 \tag{1.7}$$

where

$$\delta = \frac{|B_1|^{\alpha_1} |B_2|^{\alpha_2}}{\Gamma_p(\alpha_1) \Gamma_p(\alpha_2)}. \tag{1.8}$$

Consider the transformation $U = X_1 - X_2$ and $V = X_1$. The Jacobian is unity and the joint density of U and V, denoted by $g(U,V)$, is given by

$$g(U,V)\mathrm{d}U\mathrm{d}V = \delta|V|^{\alpha_1-\frac{p+1}{2}}|V-U|^{\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(B_1V+B_2(V-U))}\mathrm{d}U\mathrm{d}V. \tag{1.9}$$

The density of U, denoted by $g(U)$, is available by integrating out V from (1.9). Note that when U is oriented as described earlier there are only two possible situations, $U>0, V>U$ and $U<0, V>0$. Let us consider these two situations separately. For $U>0, V>U$ let

$$\begin{aligned} g_1(U) &= \int_{V>U} g(U,V)\mathrm{d}V \\ &= \delta\int_{V>U}|V|^{\alpha_1-\frac{p+1}{2}}|V-U|^{\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(B_1V+B_2(V-U))}\mathrm{d}V \\ &= \delta|U|^{\alpha_1+\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(B_1U)}\int_{Z>0}|Z|^{\alpha_2-\frac{p+1}{2}} \\ &\times |I+Z|^{\alpha_1-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}\left(U^{\frac{1}{2}}(B_1+B_2)U^{\frac{1}{2}}Z\right)}\mathrm{d}Z \end{aligned} \tag{1.10}$$

by making the transformations $Y = V-U$ and $Z = U^{-\frac{1}{2}}YU^{-\frac{1}{2}}$ for fixed U. The integral in (1.10) even when $p=1$, that is in the scalar case, is difficult. In the scalar case it can be evaluated in terms of a Whittaker function which can then be represented as a linear combination of two confluent hypergeometric functions at least in some special cases. But when $p>1$ no such representation is available due to the fact that when $p>1$

$$\int_{X=X'>0} \neq \int_{0<X=X'<1} + \int_{X=X'\geq I}$$

because $0<X<I$ implies only that the eigenvalues of X are between zero and one and $X\geq I$ gives the eigenvalues greater than or equal to one. But a whole class of matrices with some eigenvalues less than one and others greater than or equal to one are left out from the set $X=X'>0$. Hence the integral in (1.10) is not available even from a matrix-variate version of a confluent hypergeometric function. We need the Whittaker function of matrix argument to deal with this integral. This will be defined next.

2. WHITTAKER FUNCTION OF MATRIX ARGUMENT

When studying some properties of the matrix-variate Laplace transform, as applied to special functions of matrix argument, it was necessary to come up with an expression for an integral of the following type.

$$\begin{aligned} \int_{X>A}|X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X &= \mathrm{e}^{-\mathrm{tr}(A)}\int_{Y>0}|Y+A|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(Y)}\mathrm{d}Y \\ &= |A|^{\alpha}\mathrm{e}^{-\mathrm{tr}(A)}\int_{Z>0}|I+Z|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(AZ)}\mathrm{d}Z \end{aligned}$$

by making the transformations $Y = X-A, Z = A^{-\frac{1}{2}}YA^{-\frac{1}{2}}$. Hence Mathai and Pederzoli (1996) defined a function $M(\alpha,\beta;A)$ as follows:

$$M(\alpha,\beta;A) = \int_{X=X'>0}|X|^{\alpha-\frac{p+1}{2}}|I+X|^{\beta-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(AX)}\mathrm{d}X \tag{2.1}$$

for $A=A'>0, \mathrm{Re}(\alpha)>\frac{p-1}{2}$ and no restrictions on β. Then we may observe the following from the definition itself.

$$M\left(\alpha,\frac{p+1}{2};A\right) = |A|^{-\alpha}\Gamma_p(\alpha),\ \mathrm{Re}(\alpha)>\frac{p-1}{2}. \tag{2.2}$$

But for $p=1$, that is , in the scalar variable case the function $M(\cdot,\cdot;\cdot)$ is associated with a Whittaker function. Hence we define a Whittaker function in terms of $M(\cdot,\cdot;\cdot)$ as follows:

$$M(\mu,\nu;A) = |A|^{-\frac{\mu+\nu}{2}}\Gamma_p(\mu)\mathrm{e}^{\frac{1}{2}\mathrm{tr}(A)}W_{\frac{1}{2}(\nu-\mu),\frac{1}{2}(\nu+\mu-(p+1)/2)}(A) \tag{2.3}$$

for $A = A' > 0, \mathrm{Re}(\mu) > \frac{p-1}{2}$ and no restriction on ν. In terms of the integral representation we have

$$\int_{Z>0} |Z|^{\mu-\frac{p+1}{2}}|I+Z|^{\nu-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(AZ)}\mathrm{d}Z$$
$$= |A|^{-\frac{\mu+\nu}{2}}\Gamma_p(\mu)\mathrm{e}^{\frac{1}{2}\mathrm{tr}(A)}W_{\frac{1}{2}(\nu-\mu),\frac{1}{2}(\nu+\mu-(p+1)/2)}(A). \tag{2.4}$$

By using (2.4) one can establish a number of results on Whittaker function which will generalize the corresponding univariate results to the matrix-variate case. For the sake of illustration one such result will be listed here, before going back to the density in (1.10). This result follows from the above definition itself.

Theorem 2.1. *For* $\mathrm{Re}(\beta-\alpha) > \frac{p-3}{4}, A = A' > 0,$

$$\int_{Z>0} |Z|^{\beta-\alpha-\frac{p+1}{4}}|I+Z|^{\alpha+\beta-\frac{p+1}{4}}\mathrm{e}^{-\mathrm{tr}(AZ)}\mathrm{d}Z$$
$$= |A|^{-\beta-\frac{p+1}{4}}\Gamma_p\left(\beta-\alpha+\frac{p+1}{4}\right)\mathrm{e}^{\frac{1}{2}\mathrm{tr}(A)}W_{\alpha,\beta}(A).$$

This follows from the definition itself by taking $\alpha = \frac{1}{2}(\nu-\mu)$ and $\beta = \frac{1}{2}\left(\nu+\mu-\frac{p+1}{2}\right)$.

Compare the integral in (1.10) with the result in Theorem 2.1 to obtain

$$g_1(U) = \delta|B_1+B_2|^{-\frac{\alpha_1+\alpha_2}{2}}\Gamma_p(\alpha_2)h_1(U)$$

where

$$h_1(U) = \mathrm{e}^{\frac{1}{2}\mathrm{tr}((B_2-B_1)U)}|U|^{\frac{\alpha_1+\alpha_2}{2}-\frac{p+1}{2}}W_{\frac{1}{2}(\alpha_1-\alpha_2),\frac{1}{2}(\alpha_1+\alpha_2-(p+1)/2)}\left((B_1+B_2)^{\frac{1}{2}}U(B_1+B_2)^{\frac{1}{2}}\right). \tag{2.5}$$

Now for $U < 0, V > 0$ let

$$g_2(U) = \int_{V>0} g(U,V)\mathrm{d}V.$$

Write $W = -U$ so that $W = W' > 0$. After taking out W make the change $Z = W^{-\frac{1}{2}}VW^{-\frac{1}{2}}$ for fixed W to obtain

$$g_2(U) = \delta|W|^{\alpha_1+\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(B_2W)}$$
$$\times \int_{Z>0} |Z|^{\alpha_1-\frac{p+1}{2}}|I+Z|^{\alpha_2-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}\left(W^{\frac{1}{2}}(B_1+B_2)W^{\frac{1}{2}}Z\right)}\mathrm{d}Z,\ W = -U.$$

Now in the light of Theorem 2.1, we have, for $W = -U$.

$$g_2(U) = \delta|B_1+B_2|^{-\frac{\alpha_1+\alpha_2}{2}}\Gamma_p(\alpha_1)h_2(W)$$

where

$$h_2(W) = |W|^{\frac{\alpha_1+\alpha_2}{2}-\frac{p+1}{2}}\mathrm{e}^{\frac{1}{2}\mathrm{tr}((B_1-B_2)W)}W_{\frac{1}{2}(\alpha_2-\alpha_1),\frac{1}{2}(\alpha_1+\alpha_2-(p+1)/2)}\left((B_1+B_2)^{\frac{1}{2}}W(B_1+B_2)^{\frac{1}{2}}\right). \tag{2.6}$$

Hence the density of $Y = X_1 - X_2$, in the oriented case discussed in (1.10), is given by the following

Theorem 2.2.

$$g(Y) = \begin{cases} c^{-1}h_1(Y),\ Y > 0 \\ c^{-1}h_2(W),\ W = -Y > 0 \\ 0,\ \text{elsewhere,} \end{cases} \tag{2.7}$$

where $h_1(Y)$ and $h_2(W)$ are given in (2.5) and (2.6) respectively, $c = c_1 + c_2$, with

$$c_1 = \int_{Y>0} h_1(Y)\mathrm{d}Y \text{ and } c_2 = \int_{W>0} h_2(W)\mathrm{d}W.$$

The density in (2.7) generalizes the univariate case Mathai-1,4 of Mathai (1993a, p.34).

Integrals of the type in (1.10) appear in a large variety of statistical distribution problems connectd with sum, difference and linear functions of positve random variables, especially of the gamma type, or when dealing with the upper part of the incomplete gamma function. The corresponding matrix-variate analogues can be handled by using the definition of Whittaker function given above. When dealing with the distribution of a linear function of three independent real gamma variables one has to evaluate a double integral where the kernel part is of the type in (1.10). This leads to integrals involving Whittaker functions, see Mathai and Pederzoli (1996). (Distributions of linear functions of gamma variables are discussed in Mathai and Provost (1992)). Several results on integrals involving Whittaker functions of matrix argument are given by the author and his co-workers recently. For the sake of illustration one such result will be given here as a theorem.

Theorem 2.3. *For* $A = A' > 0$, $B = B' > 0$, $\mathrm{Re}\,(\gamma \pm \beta) > \frac{p-3}{4}$, $\mathrm{Re}\,(\beta - \alpha) > \frac{p-3}{4}$, $Z = Z' > 0$,

$$\begin{aligned}
&\int_{Z>0} |Z|^{\gamma-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(AZ)} W_{\alpha,\beta}(BZ)\mathrm{d}Z = \left|A + \frac{1}{2}B\right|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \\
&\times \frac{\Gamma_p\left(\gamma+\beta+\frac{p+1}{4}\right)\Gamma_p\left(\gamma-\beta+\frac{p+1}{4}\right)}{\Gamma_p\left(\gamma-\alpha+\frac{p+1}{2}\right)} \\
&\times {}_2F_1\left(\frac{p+1}{4}+\beta-\alpha,\ \frac{p+1}{4}+\gamma+\beta;\right. \\
&\left.\frac{p+1}{2}+\gamma-\alpha; I - B^{\frac{1}{2}}\left(A+\frac{1}{2}B\right)^{-1} B^{\frac{1}{2}}\right) \qquad (2.8) \\
&\qquad \textit{for } 0 < B^{\frac{1}{2}}\left(A+\frac{1}{2}B\right)^{-1} B^{\frac{1}{2}} < I \textit{ or } 2B^{-\frac{1}{2}}AB^{-\frac{1}{2}} > I \\
&= |B|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \frac{\Gamma_p\left(\frac{p+1}{4}+\gamma+\beta\right)\Gamma_p\left(\frac{p+1}{4}+\gamma-\beta\right)}{\Gamma_p\left(\frac{p+1}{2}+\gamma-\alpha\right)} \\
&\times {}_2F_1\left(\frac{p+1}{4}+\gamma-\beta, \frac{p+1}{4}+\gamma+\beta; \frac{p+1}{2}+\gamma-\alpha; -C,\right) \qquad (2.9) \\
&\qquad\qquad \textit{for } \|C\| < 1
\end{aligned}$$

where

$$C = B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - \frac{1}{2}I$$

and $\|(\cdot)\|$ *denotes a norm of* $(\cdot)$. *A sufficient condition is that*

$$0 < B^{-\frac{1}{2}}AB^{-\frac{1}{2}} - \frac{1}{2}I < I \textit{ or } 0 < \frac{1}{2}I - B^{-\frac{1}{2}}AB^{-\frac{1}{2}} < I.$$

Proof. Replace $W_{\alpha,\beta}(BX)$ by its integral representation to obtain

$$\begin{aligned}
\text{left side} &= \int_{X>0} |X|^{\gamma-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(AX)} W_{\alpha,\beta}(BX)\mathrm{d}X \\
&= \int_{X>0} \frac{|X|^{\gamma+\beta+\frac{p+1}{4}-\frac{p+1}{2}}}{\Gamma_p\left(\frac{p+1}{4}+\beta-\alpha\right)} \mathrm{e}^{-\mathrm{tr}(AX)-\frac{1}{2}\mathrm{tr}(BX)} \\
&\times \int_{Z>0} |Z|^{\beta-\alpha-\frac{p+1}{4}} |I+Z|^{\alpha+\beta-\frac{p+1}{4}} \mathrm{e}^{-\mathrm{tr}(BXZ)}\mathrm{d}Z\mathrm{d}X.
\end{aligned}$$

The X-integral can be evaluated by using a gamma integral to get

$$\begin{aligned}
&\int_{X>0} |X|^{\gamma+\beta+\frac{p+1}{4}-\frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}\left[\left(\left(A+\frac{1}{2}B\right)+BZ\right)X\right]} \mathrm{d}X \\
&\quad = \Gamma_p\left(\gamma+\beta+\frac{p+1}{4}\right)\left|\left(A+\frac{1}{2}B\right)+BZ\right|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \\
&\qquad \text{for } \mathrm{Re}(\gamma+\beta) > \frac{p-3}{4}.
\end{aligned} \tag{2.10}$$

The Z-integral, denoted by f, is given by

$$\begin{aligned}
f &= \int_{Z>0} |Z|^{\beta-\alpha-\frac{p+1}{4}} |I+Z|^{\alpha+\beta-\frac{p+1}{4}} \left|\left(A+\frac{1}{2}B\right)+BZ\right|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \mathrm{d}Z \\
&= \left|A+\frac{1}{2}B\right|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \int_{Z>0} |Z|^{\beta-\alpha-\frac{p+1}{4}} |I+Z|^{\alpha+\beta-\frac{p+1}{4}} \\
&\quad \times \left|I+B^{\frac{1}{2}}\left(A+\frac{1}{2}B\right)^{-1} B^{\frac{1}{2}} Z\right|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \mathrm{d}Z.
\end{aligned}$$

Note that in the determinant we can also replace $\left(A+\frac{1}{2}B\right)^{-1} B$ by $B^{\frac{1}{2}}\left(A+\frac{1}{2}B\right)^{-1} B^{\frac{1}{2}}$. Write $Z = (I-U)^{-\frac{1}{2}} U (I-U)^{-\frac{1}{2}} \Rightarrow I+Z^{-1} = U^{-1} \Rightarrow |Z|^{-(p+1)}\mathrm{d}Z = |U|^{-(p+1)}\mathrm{d}U \Rightarrow \mathrm{d}Z = |I-U|^{-(p+1)}\mathrm{d}U$ and $0 < U < I$. Then

$$\begin{aligned}
f &= \left|A+\frac{1}{2}B\right|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \int_{0<U<I} |U|^{\beta-\alpha-\frac{p+1}{4}} |I-U|^{\gamma-\beta-\frac{p+1}{4}} \\
&\quad \times \left|I-\left(I-B^{\frac{1}{2}}\left(A+\frac{1}{2}B\right)^{-1} B^{\frac{1}{2}}\right) U\right|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \mathrm{d}U.
\end{aligned}$$

Now we can write the integral as a ${}_2F_1$ by using Mathai (1993a, p.179). Then

$$\begin{aligned}
f &= \left|A+\frac{1}{2}B\right|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \\
&\quad \times \frac{\Gamma_p\left(\beta-\alpha+\frac{p+1}{4}\right)\Gamma_p\left(\gamma-\beta+\frac{p+1}{4}\right)}{\Gamma_p\left(\gamma-\alpha+\frac{p+1}{2}\right)} \\
&\quad \times {}_2F_1\left(\frac{p+1}{4}+\beta-\alpha, \frac{p+1}{4}+\gamma+\beta; \right. \\
&\quad \left. \frac{p+1}{2}+\gamma-\alpha; I-B^{\frac{1}{2}}\left(A+\frac{1}{2}B\right)^{-1} B^{\frac{1}{2}}\right)
\end{aligned}$$

for $0 < I - B^{\frac{1}{2}}\left(A+\frac{1}{2}B\right)^{-1} B^{\frac{1}{2}} < I \Rightarrow 0 < B^{\frac{1}{2}}\left(A+\frac{1}{2}B\right)^{-1} B^{\frac{1}{2}} < I \Rightarrow 2B^{-\frac{1}{2}}AB^{-\frac{1}{2}} > I$. Substituting back, one gamma is cancelled, we get (2.8). But observe that

$$\begin{aligned}
\left|\left(A+\frac{1}{2}B\right)+BZ\right| &= \left|\left(A-\frac{1}{2}B\right)+B(I-Z)\right| \\
&= |B|\,|I-Z|\,\left|I+\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}-\frac{1}{2}I\right)(I+Z)^{-1}\right|.
\end{aligned}$$

Now

$$f = |B|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \int_{Z>0} |Z|^{\beta-\alpha-\frac{p+1}{4}} |I+Z|^{\alpha-\gamma-\frac{p+1}{2}}$$
$$\times \left| I + \left(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} - \frac{1}{2} I \right) (I+Z)^{-1} \right|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \mathrm{d}Z.$$

Put $U = (I+Z)^{-1} \Rightarrow \mathrm{d}Z = |U|^{-(p+1)}\mathrm{d}U$, $|Z| = |U|^{-1}|I-U|, |I+Z| = |U|^{-1}, 0 < U < I$. Then

$$f = |B|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \int_{0<U<I} |U|^{\gamma-\beta-\frac{p+1}{4}} |I-U|^{\beta-\alpha-\frac{p+1}{4}} |I+CU|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)}$$

where

$$C = B^{-\frac{1}{2}} A B^{-\frac{1}{2}} - \frac{1}{2} I.$$

Evaluating this integral by using Mathai (1993a, p.179) one has

$$f = |B|^{-\left(\gamma+\beta+\frac{p+1}{4}\right)} \frac{\Gamma_p\left(\frac{p+1}{4}+\gamma-\beta\right)\Gamma_p\left(\frac{p+1}{4}+\beta-\alpha\right)}{\Gamma_p\left(\frac{p+1}{2}+\gamma-\alpha\right)}$$
$$\times {}_2F_1\left(\frac{p+1}{4}+\gamma-\beta, \gamma+\beta+\frac{p+1}{4}; \frac{p+1}{2}+\gamma-\alpha; -C\right),$$
$$\text{for } \|C\| < 1,\ \mathrm{Re}\,(\gamma-\beta) > \frac{p-3}{4}, \mathrm{Re}\,(\beta-\alpha) > \frac{p-3}{4}.$$

Substituting back we get (2.9), noting that one gamma is cancelled.

3. RESIDUAL VARIABLES IN THE COMPLEX CASE

Let $\tilde{X}_1$ and $\tilde{X}_2$ be $p \times p$ hermitian positive definite matrix random variables defined in the complex field. Such a matrix $\tilde{X}_j$ can be expressed in the following form:

$$\tilde{X}_j = X_{j1} + iX_{j2}$$

where $i = \sqrt{-1}$, X_{j1} and X_{j2} are matrices with real elements such that $X_{j1} = X_{j2}' > 0$ and $X_{j2}' = -X_{j2}$. That is, X_{j1} is symmetric positive definite and X_{j2} is skew symmetric. All the matrices appearing in this section are $p \times p$ hermitian positive definite unless stated otherwise.

If the growth and decay matrices in the complex field are respectively $\tilde{X}_1$ and $\tilde{X}_2$ then the residual variable is

$$\tilde{Y} = \tilde{X}_1 - \tilde{X}_2. \tag{3.1}$$

Let $\tilde{X}_1$ and $\tilde{X}_2$ be independently distributed matrix-variate complex gamma variables with the densities

$$\tilde{g}_j(\tilde{X}_j) = \frac{|\det(\tilde{B}_j)|^{\alpha_j}}{\tilde{\Gamma}_p(\alpha_j)} |\det(\tilde{X}_j)|^{\alpha_j-p} \mathrm{e}^{-\mathrm{tr}(\tilde{B}_j\tilde{X}_j)},\ \tilde{X}_j = \tilde{X}_j^* > 0,\ \mathrm{Re}(\alpha_j) > p-1 \tag{3.2}$$

and $\tilde{g}_j(\tilde{X}_j) = 0$ elsewhere, where $(\cdot)^*$ denotes the conjugate transpose of $(\cdot)$, $\tilde{X}_j = \tilde{X}_j^* > 0$ implies that $\tilde{X}_j$ is hermitian positive definite, $\det(\cdot)$ denotes the determinant of $(\cdot)$, $|\det(\cdot)|$ gives the absolute value of the determinant and $\tilde{\Gamma}_p(\cdot)$ is the matridx-variate gamma in the complex case, given by

$$\tilde{\Gamma}_p(\alpha) = \int_{\tilde{X}=\tilde{X}^*>0} |\det(\tilde{X})|^{\alpha-p} \mathrm{e}^{-\mathrm{tr}(\tilde{X})} \mathrm{d}\tilde{X} \tag{3.3}$$
$$= \pi^{p(p-1)/2}\Gamma(\alpha)\Gamma(\alpha-1)...\Gamma(\alpha-p+1),\ \mathrm{Re}(\alpha) > p-1.$$

Quadratic forms in complex Gaussian random vectors, giving rise to Whittaker variables, are used in a wide variety of problems in communication and engineering areas, see for example, Hirasawa (1988), Divsalar et al.(1990), and Biyari and Lindsey (1991).

Our aim here is to look at the residual variable in (3.1) when the input and output variables are independently distributed with the densities in (3.2). Then proceeding as in (1.7) to (1.10) we note that in order to compute the density of the residual variable one needs an integral to be evaluated, corresponding to the one in (1.10). This requires the definition of a Whittaker function of matrix argument in the complex case. Whittaker function of matrix argument in the complex case, denoted by $\tilde{W}_{\cdot,\cdot}(\cdot)$, will be defined as that symmetric function, symmetric in the sense

$$\tilde{W}_{\alpha,\beta}(\tilde{A}\tilde{B}) = \tilde{W}_{\alpha,\beta}(\tilde{B}\tilde{A})$$

for $p \times p$ hermitian positive definite matrices, having the following integral representaton.

$$\begin{aligned}\int_{\tilde{Z}=\tilde{Z}^*>0} &|\det(\tilde{Z})|^{\beta-\alpha-\frac{p}{2}}|\det(I+\tilde{Z})|^{\beta+\alpha-\frac{p}{2}}\ \mathrm{e}^{-\mathrm{tr}(\tilde{A}\tilde{Z})}\mathrm{d}\tilde{Z}\\ &= |\det(\tilde{A})|^{-\beta-\frac{p}{2}}\ \tilde{\Gamma}_p\left(\beta-\alpha+\frac{p}{2}\right)\mathrm{e}^{\frac{1}{2}\mathrm{tr}(\tilde{A})}\tilde{W}_{\alpha,\beta}(\tilde{A}) \qquad (3.4)\end{aligned}$$

for $\mathrm{Re}(\beta-\alpha) > \frac{p}{2}-1$.

As a consequence of the definition itself we can have the following result which will be stated as a theorem.

Theorem 3.1. *For* $\tilde{A} = \tilde{A}^* > 0$

$$\begin{aligned}\int_{\tilde{Z}=\tilde{Z}^*>0} &|\det(I+\tilde{Z})|^{-\alpha}\mathrm{e}^{-\mathrm{tr}(\tilde{A}\tilde{Z})}\mathrm{d}\tilde{Z}\\ &= |\det(\tilde{A})|^{\frac{\alpha}{2}-p}\ \tilde{\Gamma}_p(p)\mathrm{e}^{\frac{1}{2}\mathrm{tr}(\tilde{A})}\tilde{W}_{-\frac{\alpha}{2},\frac{1}{2}(-\alpha+p)}(\tilde{A}).\end{aligned}$$

One can establish a number of results on Whittaker function in the complex case which will be useful in working out the distributions of sums, differences and linear functions of matrix gamma variables in the complex case, evaluating probabilities associated with these distributions, evaluating the incomplete gamma, incomplete type-2 beta and other related integrals, and related problems. In order to illustrate the techniques one more result will be given here.

Theorem 3.2. *For* $\tilde{B} = \tilde{B}^* > 0,\ \tilde{U} = \tilde{U}^* > 0,\ \tilde{M} = \tilde{M}^* > 0,\ \mathrm{Re}(q) > \frac{p}{2}$,

$$\begin{aligned}\int_{\tilde{X}>\tilde{U}} &|\det(\tilde{X}+\tilde{B})|^{2\alpha-p}|\det(\tilde{X}-\tilde{U})|^{2q-p}\mathrm{e}^{-\mathrm{tr}(\tilde{M}\tilde{X})}\mathrm{d}\tilde{X}\\ &= \tilde{\Gamma}_p(2q)|\det(\tilde{U}+\tilde{B})|^{\alpha+q-p}|\det(\tilde{M})|^{-(\alpha+q)}\\ &\times \mathrm{e}^{\frac{1}{2}\mathrm{tr}[(\tilde{B}-\tilde{U})\tilde{M}]}\tilde{W}_{(\alpha-q),(\alpha+q-\frac{p}{2})}(\tilde{T}),\\ \tilde{T} &= (\tilde{U}+\tilde{B})^{\frac{1}{2}}\tilde{M}(\tilde{U}+\tilde{B})^{\frac{1}{2}}.\end{aligned}$$

Proof. Consider the following transformations.

$$\begin{aligned}\tilde{Y} &= \tilde{X}-\tilde{U},\ \tilde{Z} = (\tilde{B}+\tilde{U})^{-\frac{1}{2}}\tilde{Y}(\tilde{B}+\tilde{U})^{-\frac{1}{2}}\\ &\Rightarrow \mathrm{d}\tilde{Z} = |\det(\tilde{B}+\tilde{U})|^{-p}\mathrm{d}\tilde{Y}.\end{aligned}$$

Now the integral on the left side, denoted by f, is given by the following:

$$\begin{aligned}f &= \mathrm{e}^{-\mathrm{tr}(\tilde{M}\tilde{U})}|\det(\tilde{B}+\tilde{U})|^{2\alpha+2q-p}\\ &\times \int_{\tilde{Z}=\tilde{Z}^*>0}|\det(\tilde{Z})|^{2q-p}|\det(I+\tilde{Z})|^{2\alpha-p}\\ &\times \exp\left\{-\mathrm{tr}\left[\tilde{M}(\tilde{B}+\tilde{U})^{\frac{1}{2}}\tilde{Z}(\tilde{B}+\tilde{U})^{\frac{1}{2}}\right]\right\}\mathrm{d}\tilde{Z}\end{aligned}$$

for $\mathrm{Re}(2q) > p - 1$. Now interpreting with the help of the definition of a Whittaker function the result follows.

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