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LOGNORMAL RETURNS, EFFICIENT FRONTIER, AND SHORTFALL CONSTRAINT

Luca Ghezzi[*]

1. Introduction

Whether portfolio management is active or passive, the notion of **efficient frontier** comes in useful when it comes to trading off risk and return within a process of **strategic asset allocation**. Active management and passive management are contrasted in Section 2, where the stages of an investment process are also recalled. Reference is usually made to an aggregate portfolio that is rebalanced annually to restore the weights of its strategic asset allocation. To derive an efficient frontier,

- each feasible portfolio, i.e. each feasible set of weights, is represented by a mean and a standard deviation;
- mean-variance optimisation is performed. Inputs include the **simple returns** of all asset classes, i.e. their means and covariances, whereas outputs are the **efficient portfolios**, i.e. the optimal sets of weights.

However, the use of simple returns and arithmetic means clashes with the calculation of both a long-term forecast accumulation and its confidence interval. As remarked by MacBeth (1995), such a problem is important in practice. As explained in Section 3 below, it can be overcome by the use of logarithmic returns and geometric means in place of simple returns and arithmetic means.

Unfortunately, there are no exact formulas that turn the **population** or **sample** moments (of logarithmic returns) of all asset classes into the two portfolio moments needed, i.e. the two moments of a strategic asset allocation. A solution is pointed out in Elton and Gruber (1974), where **portfolio** returns are assumed to be **lognormally** distributed. More precisely, a mapping is considered from a usual efficient frontier based on simple returns to an alternative efficient frontier based on logarithmic returns. With the benefit of hindsight we can claim that the analysis is incomplete owing to an insufficient characterisation of the usual efficient frontier.

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This paper expands on Elton and Gruber (1974) as well as Tobin (1958), Black (1972), and Merton (1972). It provides a more comprehensive treatment, focusing on

- how the population moments of simple returns turn into the population moments of logarithmic returns, whatever the feasible porfolio;
- how efficient portfolios based on simple returns may turn into efficient portfolios based on logarithmic returns.

On the one hand, additional theoretical properties of the mapping are readily obtained. On the other one, the mapping is applied to two different data sets, i.e. the annual total returns of three major US asset classes over the period 1926-1997 as well as the annual total returns of four equity classes over the period 1972-2005. For the sake of clarity, passive management is specifically considered; nonetheless, our theoretical results have a more general validity within portfolio theory. The analysis is carried out under the usual simplifying assumption that the annual total returns of all feasible portfolios are time incorrelated and stationary, population moments being the same as historical moments. Although this clashes with the empirical evidence, it makes the optimisation model both analytically tractable and operationally useful, especially to financial advisors and institutional investors.

This paper extends the literature on portfolio selection by enabling an investor to obtain a better estimate of the trade-off between risk and return; more precisely, each efficient portfolio is matched with a mean return and a threshold return along with its confidence level. Different time horizons can be considered using only annual moments; moreover, each threshold return can be turned into a threshold accumulation that has the same confidence level. *Coeteris paribus*, the more distant the time horizon, the higher the confidence level.

The paper is organised as follows. An insightful problem of passive management is introduced in Section 2, where both data sets are also presented. The properties of **logarithmic returns** are recalled in Section 3 in connection with the above-mentioned problem of passive management. **Simple returns** are considered in Section 4, where mean-variance optimisation is performed thus obtaining an efficient frontier. In doing so, the relevant properties of the efficient frontier are listed. Next, an appropriate mapping is presented in Section 5, where the usual efficient frontier is turned into an alternative efficient frontier based on **logarithmic returns**. In doing so, new properties of the mapping are derived. Discussion and conclusions are provided in Section 6, where the implications of the mapping are examined. More precisely, the shortfall constraint approach by Leibowitz and Henriksson (1989) is extended to the case of lognormal rather than normal portfolio returns.

2. Application and data sets

Portfolio management is a process made up of three main stages: strategic asset allocation, tactical asset allocation, and asset selection (see Farrell, 1997, pp. 231-233). **Strategic asset allocation** concerns the determination of all asset classes and their percent target weights, i.e. of an aggregate portfolio. Its performance is considerably affected by the risk appetite; in the long run, a larger and reasonable risk is usually matched by a larger average return (see Siegel, 2014, chapt. 5). A well crafted portfolio includes 8-10 asset classes at most. **Tactical asset allocation** regards slight opportunistic changes of the percent target weights on the basis of (qualitative) short run forecasts. Asset selection is about picking single assets within an asset class. Tactical asset allocation and asset selection are periodically reviewed by asset managers. Some successful investors, including Warren Buffett, are in favour of **passive neglect**, which implies a slow turnover, say once a year at most, possibly in response to important economic news.

Individual and institutional investors as well as financial advisors, security analysts, and portfolio managers take part in the above-mentioned process; institutional investors also include pension funds, endowments (e.g. of Anglo-Saxon universities), and foundations (e.g. Italian banking ones).

Portfolio management can be either **active** or **passive**. Active management focuses on trend timing and mispriced asset selection, two very challenging activities indeed. As for equities, US security analysts and portfolio managers have proved to be more at ease with mispriced asset selection than trend timing (see Damodaran, 2003, p. 393). In contrast, passive management gives up both tactical asset allocation and asset selection. It makes use of index mutual funds and exchange traded funds, which track their benchmarks; they are regularly rebalanced to restore the percent weights of the strategic asset allocation. Therefore, money is reallocated from the asset classes that have recently performed better to the asset classes that have recently performed worse. Passive management is consistent with the efficient market hypothesis, whereby the persistent outperformance of a benchmark is very unlikely. Elton *et al.* (2010, chapt. 17) provide an educated summary of the whopping body of empirical evidence in favour of and against the efficient market hypothesis. According to such an empirical evidence, the managers of mutual funds as a whole are usually unable to cover their expenses with extra returns: therefore most mutual funds, be they bond or equity funds, usually underperform their benchmarks.

In the following, reference is made to two different aggregate portfolios. The first one includes three major US asset classes: short-term Treasury bills with a maturity of one month,

long-term Treasury bonds with an average maturity of 20 years, and stocks. All coupons and dividends are assumed to be reinvested. The data set is made up of annual nonoverlapping **total** returns. It spans a time period that runs from the beginning of 1926 to the end of 1997. More precisely, the Ibbotson Associates short-term Treasury bill index, the Ibbotson Associates long-term Treasury bond index, and the S&P 500 stock index are considered along with the Consumer Price Index. All data are reported in Cornell (1999, chapt. 1); they were gathered by Ibbotson Associates, a well known information provider.

The sample moments of all three major US asset classes are reported in Table 1; **simple returns** r are considered in Table 1a, whereas **logarithmic returns** $\ln(1+r)$ are considered in Table 1b. The Consumer Price Index (CPI) is treated accordingly. Notice that mean logarithmic returns are lower than mean simple returns. As remarked in Section 3, this is true in general.

Table 1a – Sample moments of three major US asset classes: **annual simple returns**, 1926-1997

	Mean return %	**Standard deviation %**	**Correlation matrix**		
			Bills	**Bonds**	**Stocks**
T-bills	3.80	3.24	1		
T-bonds	5.61	9.23	0.24	1	
Stocks	12.96	20.32	-0.03	0.19	1
CPI	3.20	4.51			

Table 1b – Sample moments of 3 major US asset classes: **annual logarithmic returns**, 1926-1997

	Mean return %	**Standard deviation %**	**Correlation**		
			Bills	**Bonds**	**Stocks**
T-bills	3.68	3.08	1		
T-bonds	5.11	8.33	0.23	1	
Stocks	10.43	19.40	0.00	0.20	1
CPI	3.06	4.38			

The second portfolio includes four major equity classes: US stocks (S&P 500 index), international stocks (MSCI EAFE index), real estate securities (FTSE NAREIT Equity REITS index), and commodity-linked securities (S&P GSCI index). The data set, reported in Gibson (2008, chapt. 12), is made up of annual nonoverlapping **total** returns for the years 1972-2005.

For the sake of simplicity, we assume that our **population moments** are the same as the sample moments of Table 1a. However, when drawing Figure 4 in the sequel, we assume that our **population moments** are the same as the sample moments of the latter data set. A more sophisticated assumption would call for estimating the average **real** rates of return and adding a forecast of the inflation rate to them. As remarked by Bernstein (1997), the data sample should begin and end with similar financial multiples, e.g. similar price-earnings ratios. Alternatively, the data sample might be split into a sequence of settings, including deflation (1929-1938),

sound growth (1946-1965), inflation (1966-1980), disinflation (1981-1997). Accordingly, reference would solely be made to the most significant settings. Moreover, those settings might be turned into 5-year scenarios, each having a specific probability of occurrence (Farrell,1997, chapt. 9).

3. Passive management, compound return, and logarithmic return

As shown by Table 1 above, stocks are riskier but eventually more rewarding than long-term Treasury bonds and short-term Treasury bills; a similar conclusion can be drawn from the wider historical summary by Siegel (2014, chapt. 5). Therefore, if the time horizon were far enough (i.e. after more than 5 years), also stocks should be chosen. Moreover, the risk tolerance of an individual investor usually improves as time goes by and he/she becomes more aware of the investment process (Gibson, 2008, chapt. 6). Nonetheless, even if capital growth were the target, all three asset classes should be considered to benefit from the risk-mitigating effect of a broad diversification.

When determing all percent target weights of an aggregate portfolio, an insightful problem of passive management may be taken into consideration. Let time be measured in years. Suppose that capital is invested in a portfolio that is **rebalanced annually** to restore the weights of its **strategic asset allocation**. Suppose that the annual total returns r of all feasible portfolios are time incorrelated and stationary, which is a reasonable and usual choice (MacBeth, 1995). Disregard commissions, fees, and personal taxes. Our setting is consistent with business practice, as reported by Gibson (2008). In his opinion, a satisfactory portfolio performance follows mostly from a sound strategic asset allocation. Undoubtedly, this is true for an investor who is not fairly experienced or has no flair for advanced portfolio management, which is possibly based on trend timing and/or mispriced asset selection.

If a forward looking analysis is performed over a future time period, a long-term forecast accumulation can be attributed to each candidate portfolio. Moreover, as explained in Section 6, a **confidence interval** can be estimated with an appropriate **confidence level**. The forecast accumulation only depends on the population mean of $\ln(1+r)$, whereas the confidence interval depends on the population mean and standard deviation of $\ln(1+r)$. In contrast, if a backward looking analysis is performed over a historical period of n years, each actual accumulation under examination is

$$\text{final accumulation} = \text{initial capital} \cdot e^{\hat{m}\, n} \tag{1}$$

where $\hat{m}$ is the sample mean of $\ln(1+r)$. Needless to say, different aggregate portfolios, i.e. different strategic asset allocations, are matched by different sample means $\hat{m}$. In other words, as annual total returns are compounded, logarithmic returns $\ln(1+r)$ must replace simple returns *r*. We have

$$\hat{m} \le e^{\hat{m}} - 1 \quad \text{as well as} \quad e^{\hat{m}} - 1 \le \hat{\mu}$$

where $\hat{\mu}$ and $e^{\hat{m}} - 1$ are the sample means of simple returns *r*, **arithmetic** and **geometric** respectively. The former inequality follows from Maclaurin's expansion truncated at first order; it is strict, unless all simple (and hence logarithmic) returns are equal to zero. The latter inequality follows from Jensens's inequality; it is strict, unless all simple (and hence logarithmic) returns are constant. Population means share both properties.

Unfortunately, the computation of a mean logarithmic return and the accompanying variance is not straightforward. No approximation is needed to compute the logarithmic mean returns and (co)variances for all asset classes. Nonetheless, there are no exact formulas that turn the **sample** (or **population**) moments of all asset classes into the two portfolio moments. Actually, exact formulas are available under the assumption that the portfolio is continuously rebalanced; see the textbook Luenberger (1998, chapt. 15) as well as the more advanced treatment of Fernholz and Shay (1982). However, such an ideal assumption overlooks the role played by commissions, fees, and taxes. Moreover, it clashes with the notion of annual rebalancing, which rests on a compromise between the exposure to risk and a slow turnover.

An alternative approach rests on the assumption that portfolio returns are lognormally distributed (Elton and Gruber, 1974). Such an assumption doesn't clash with the notion of annual rebalancing. More precisely, **simple returns** *r* can be used at first. Performing portfolio optimisation obtains an **efficient frontier**. This is shown in Section 4. Next, some properties of the lognormal distribution can be considered and an appropriate mapping can be derived. As a consequence, the usual efficient frontier is turned into an alternative efficient frontier based on **logarithmic returns** $\ln(1+r)$. This is done in Section 5, where the above-mentioned mapping is presented, updated, and applied to appropriate historical data. Note that the mapping only applies to **population** moments.

4. Simple returns and the usual efficient frontier

Let time be measured in years. Recall that capital is invested into a portfolio that is **rebalanced annually** to restore the weights of its **strategic asset allocation**. The annual total

returns of each feasible portfolio are time incorrelated and stationary. Commissions, fees, and personal taxes are disregarded.

In our setting, each feasible asset allocation, i.e. each feasible set of weights, is represented by a mean return μ and a standard deviation σ; the former measures return, whereas the latter measures risk. As each candidate asset allocation must be an efficient portfolio, the selection process is carried out in two stages. First, mean-variance optimisation is performed and the efficient frontier is detected. As a consequence, each feasible mean return μ (on the vertical axis) is matched by the lowest possible standard deviation σ (on the horizontal axis). Next, the most appropriate efficient portfolio is selected in line with the investor's risk tolerance. As short selling, and hence negative weights, are allowed in our setting, the efficient frontier is a (portion of a) **hyperbola**. If weights are assumed to be nonnegative, the efficient frontier remains upward sloping and strictly concave (Rudolf, 1994, chapt. 1). Therefore, the analysis of Section 5 might carry over to such an extended setting, which is more realistic but more challenging than the present one.

More precisely, our **linear-quadratic** problem takes the form

$$\min_{w_1, w_2, \cdots} \sigma^2 = \min_{w_1, w_2, \cdots} \sum_j \sum_k w_j w_k \sigma_{jk} \tag{2}$$

$$\text{subject to} \quad \sum_j w_j \mu_j = \mu \quad \text{and} \quad \sum_j w_j = 1 \tag{3}$$

where w_j is the weight of the *j*-th asset class, whereas the means μ_j and covariances σ_{jk} are taken from the Table 1a of **simple returns** *r* by way of illustration. In principle, the **linear-quadratic** problem (2)-(3) might be solved numerically for different values of μ so obtaining an array of efficient portfolios. However, mean-variance optimisation is facilitated by the **one-fund** theorem, obtained by Tobin (1958), and the **two-fund** theorem, obtained by Black (1972). Both theorems were not considered in Elton and Gruber (1974), our main reference. They are restated in the textbook by Luenberger (1998, chapt. 6) within the realm of portfolio theory.

Actually, Merton (1972) was not considered either; he showed that both the minimum-variance set and the efficient frontier can be derived analytically and obtained a **parabola** in $(\mu; \sigma^2)$ space as well as a **hyperbola** in $(\mu; \sigma)$ space, i.e.

$$\sigma^2 = \frac{a}{d}\mu^2 - \frac{2b}{d}\mu + \frac{c}{d} > 0 \quad \text{and} \quad \sigma = \sqrt{\frac{a}{d}\mu^2 - \frac{2b}{d}\mu + \frac{c}{d}} \tag{4}$$

the 4 real parameters being

$$a = \mathrm{I}^T \Sigma^{-1} \mathrm{I} > 0; \;\; b = \mathrm{M}^T \Sigma^{-1} \mathrm{I}; \;\; c = \mathrm{M}^T \Sigma^{-1} \mathrm{M} > 0; \;\; d = ac - b^2 > 0 \tag{5}$$

where I is a **unit** vector, $\mathrm{M} = \lfloor \mu_j \rfloor$ is a mean vector, and $\Sigma = \lfloor \sigma_{jk} \rfloor$ is a variance-covariance matrix. The four parameters *a*, *b*, *c*, and *d* in Equation (5) take the above-mentioned signs under the assumption that the real and symmetric matrix $\Sigma = \lfloor \sigma_{jk} \rfloor$ is **positive definite**, namely that all its eigenvalues and leading principal minors are positive. This implies that no asset class can be perfectly correlated with a portfolio made up of the remaining asset classes. In the case under examination, based on Table 1a, the four parameters are such that

$$a = 992.93; \;\; b = 40.63; \;\; c = 1.87; \;\; d = 205.84$$

Moreover, we have (see also Rudolf, 1994, chapt. 1)

$$W = \Sigma^{-1}\mathrm{M}\left(\frac{a\mu - b}{d}\right) - \Sigma^{-1}\mathrm{I}\left(\frac{b\mu - c}{d}\right) \tag{6}$$

where $W = \lfloor w_j \rfloor$ is a vector including the weights of the efficient portfolio with mean μ and hence standard deviation σ given by Equation (4).

Both our minimum-variance set and efficient frontier are portrayed in Figure 1, a risk-return diagram. The **efficient frontier** is the upper portion of the minimum-variance set.

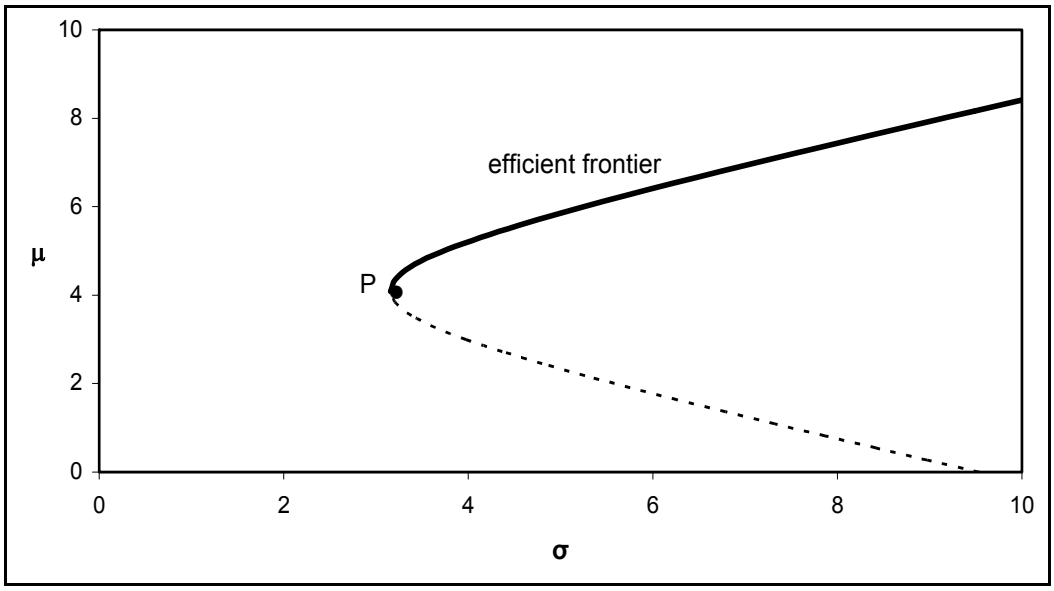

Figure 1 – Simple returns and the usual efficient frontier

The efficient frontier starts from the minimum-variance portfolio *P*, where the standard deviation takes its lowest possible value. As a larger return comes along with a larger risk, no efficient portfolio is dominant. Indeed, the investor selects the most appropriate efficient portfolio in line with his/her risk tolerance. The minimum-variance portfolio *P* is such that

$$\sigma_P = \sqrt{\frac{1}{a}} = 3.1735\% \quad \text{and} \quad \mu_P = \frac{b}{a} = 4.09\% \tag{7}$$

5. Logarithmic returns and the alternative efficient frontier

In Elton and Gruber (1974) **portfolio** returns are assumed to be lognormally distributed. In other words, every efficient portfolio given by Equation (6) is such that its **population** moments (4) obey the following equations, reported in Crow–Shimizu (1988)

$$1 + \mu = e^{m+0.5s^2} \tag{8}$$

$$\sigma^2 = (1+\mu)^2\left(e^{s^2} - 1\right) \tag{9}$$

where μ and σ are the mean and standard deviation of **simple returns** *r*, whereas *m* and *s* are the mean and standard deviation of **logarithmic returns** $\ln(1+r)$. Equations (8) and (9) represent a one-to-one mapping between points in $(s;m)$ space and points in $(\sigma;\mu)$ space.

To gain an insight into Equations (8) and (9), let's suppose that *s* is given. According to (9), any given *s* maps into the straight line

$$\mu = \frac{1}{\sqrt{e^{s^2} - 1}}\sigma - 1 \tag{10}$$

that goes through the point $(0;-1)$ in $(\sigma;\mu)$ space; the higher *s*, the lower is the gradient.

According to Equation (8), the highest (lowest) possible *m*

$$m = \ln(1+\mu) - 0.5s^2 \tag{11}$$

is matched by the highest (lowest) possible μ. As a consequence, the efficient portfolio *T* in Figure 2 maps into the minimum-variance point *T* in $(s;m)$ space.

Remark 1. In the case under examination, based on Table 1a, the efficient portfolio T is such that

$$\sigma_T = 3.1738\% \quad \text{and} \quad \mu_T = 4.11\% \tag{12}$$

namely it is very close to the minimum-variance portfolio given by Equation (7). The efficient portfolio T can be determined by applying the **one-fund** theorem, as restated by Luenberger (1998, sect. 6.9). In other words, we work out the system of linear equations

$$\begin{bmatrix} \mu_1 - r_f \\ \mu_2 - r_f \\ \mu_3 - r_f \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_1^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

where r_f is a fictitious safe rate of interest such that $(0; r_f) = (0; -1)$. Keep in mind that if the variance-covariance matrix Σ is **positive definite**, it is **invertible** as well. Next, we normalise its solution

$$w_j = \frac{z_j}{z_1 + z_2 + z_3} \quad \text{for } j = 1,2,3$$

so obtaining the three unknown weights and eventually the moments of Equation (12). More generally, as both s and the gradient of Equation (10) are positive, every efficient portfolio T must lie above the minimum-variance portfolio P.

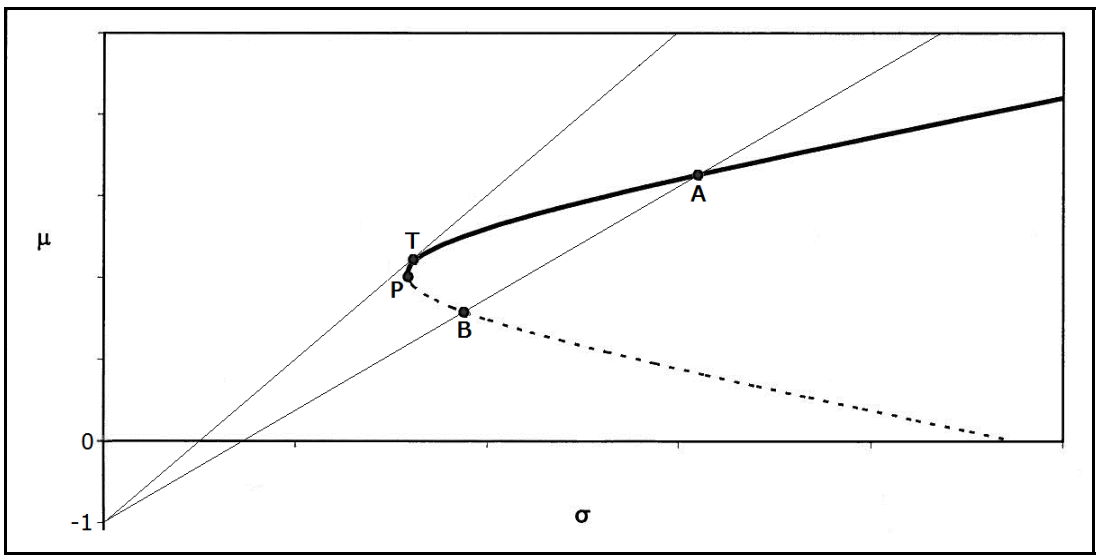

Figure 2 – Geometric properties of the mapping represented by Equations (8)-(9)

As proved below, the efficient portfolio *A* of Figure 2 may map into an efficient portfolio, whereas the minimum-variance portfolio *B* maps into a minimum-variance portfolio. As Elton and Gruber (1974) didn't take the mapping of portfolio *B* into account, they were unable to realise that as long as short selling is allowed, inefficient portfolios in $(\sigma;\mu)$ space can't map into efficient portfolios in $(s;m)$ space. To do so, consider all portfolios lying on the straight line that connects *A* and *B*. Although each portfolio has the same standard deviation *s* owing to (10), it has a lower mean return *m* compared to portfolio *A* owing to Equation (11).

Let us now state Propositions 1 and 2.

Proposition 1. All efficient or minimum-variance portfolios below point *T* of Figure 2 map into minimum-variance portfolios in $(s;m)$ space.

Proof. Reconsider Figure 2 and all portfolios lying on the straight line connecting *A* and *B* in $(\sigma;\mu)$ space. Recall that *s* is given, *m* being the lowest for portfolio *B* and the highest for portfolio *A*. Now let *s* vary, with the straight line connecting *A* and *B* rotating clockwise around the point $(0;-1)$. All efficient or minimum-variance portfolios below point *T* are such that the lower μ, the lower is the gradient of (10). Therefore, the lower μ, the higher *s* owing to Equation (10) and the lower *m* owing to Equation (11).

Proposition 2. If the discriminant

$$\Delta = (a+3b-2d)^2 - 4(a+d)(b+2c+d) \tag{13}$$

is negative, all efficient portfolios above point *T* in Figure 2 map into efficient portfolios in $(s;m)$ space.

Proof. Solving Equation (9) for s^2 and substituting s^2 into Equation (8) obtains

$$m = \ln(1+\mu) - 0.5s^2 = \ln(1+\mu) - 0.5\ln\left[1+\frac{\sigma^2}{(1+\mu)^2}\right]$$

Differentiating such equation with respect to μ obtains

$$\frac{dm}{d\mu}=\frac{1}{1+\mu}-0.5\frac{1}{1+\frac{\sigma^2}{(1+\mu)^2}}\frac{\frac{d\sigma^2}{d\mu}(1+\mu)-2\sigma^2}{(1+\mu)^3} \tag{14}$$

so that the first derivative (14) is positive whenever

$$\frac{1}{1+\mu}\left[1-0.5\frac{\frac{d\sigma^2}{d\mu}(1+\mu)-2\sigma^2}{(1+\mu)^2+\sigma^2}\right]>0$$

i.e. whenever

$$(1+\mu)^2+2\sigma^2>0.5\frac{d\sigma^2}{d\mu}(1+\mu) \tag{15}$$

Reconsider Figure 2 and all portfolios lying on the straight line connecting *A* and *B* in $(\sigma;\mu)$ space. Recall that *s* is given, *m* being lowest in B and highest in A.

Now let *s* vary, with the straight line through A and B rotating clockwise around the point $(0;-1)$. Focus on all efficient portfolios that lie above point *T*. We already know that the higher μ, the lower the gradient of (10) and the higher *s*. We want to prove that the higher μ, the higher *m*; in other words, we want to prove that the first derivative (14) is positive, i.e. that the inequality (15) is met. Substituting Equation (4) into (15) and rearranging obtains

$$(a+d)\mu^2+(-a-3b+2d)\mu+b+2c+d>0$$

The left-hand side of such inequality is a function of μ; more precisely, it is an upward bending parabola owing to Equation (5). Therefore, it doesn't cross the horizontal axis, if and only if the discriminant (13) is negative.

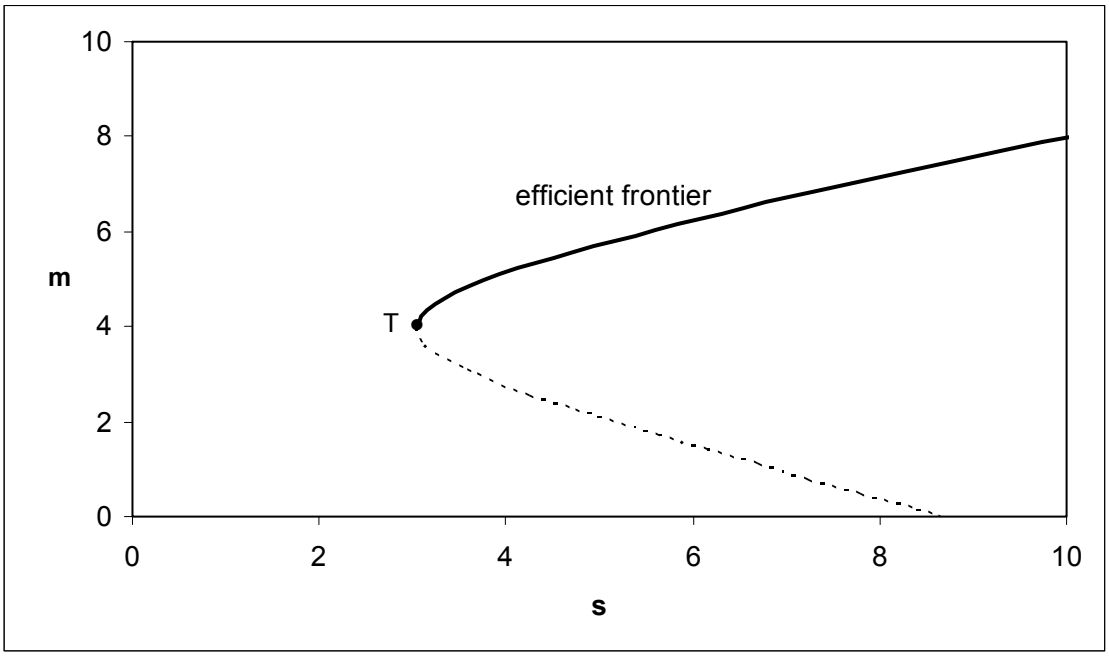

Figure 3 – Logarithmic returns and the alternative efficient frontier

In the case under examination, we have

$$\Delta = -705{,}365.35$$

Therefore, applying the mapping (8) and (9) to the usual efficient frontier of Figure 1 obtains the alternative efficient frontier plotted in Figure 3; the former is based on Table 1a and portrayed in $(\sigma;\mu)$ space, whereas the latter is portrayed in $(s;m)$ space. Remarkably, both minimum-variance sets have a similar shape.

The two efficient frontiers are compared in Table 2 as well. It is readily ascertained that both differences $\mu - m$ and $\sigma - s$ increase as μ increases. However, it is also readily realised that both ratios $\frac{\mu}{\sigma}$ and $\frac{m}{s}$ decrease as μ increases. Notice that $w_1 = -0.28$ for $\mu = 12.00\%$ and $\sigma = 17.66\%$, which implies a short position in Treasury bills.

Table 2 – Population moments of 5 efficient portfolios

w_1	w_2	w_3	μ	σ	$\boldsymbol{m}$	$\boldsymbol{s}$	μ/σ	m/s
0.96	0.02	0.02	4.00%	3.18%	3.88%	3.06%	1.26	1.27
0.65	0.14	0.21	6.00%	5.26%	5.70%	4.96%	1.14	1.15
0.34	0.25	0.41	8.00%	9.15%	7.34%	8.46%	0.87	0.87
0.03	0.37	0.60	10.00%	13.36%	8.80%	12.10%	0.75	0.73
-0.28	0.48	0.80	12.00%	17.66%	10.11%	15.67%	0.68	0.65

Finally, let us state Proposition 3.

Proposition 3. If the discriminant (13) is positive, some efficient portfolios above point *T* of Figure 2 map into inefficient portfolios in $(s;m)$ space.

Proof. Reconsider Figure 2 and focus on all efficient portfolios that lie above point *T*. We already know that the higher μ, the lower the gradient of Equation (10) and the higher *s*. We want to find the conditions under which the higher μ, the lower *m*; this amounts to finding the conditions under which the first derivative (14) is negative. Substituting (4) into (14) and rearranging obtains

$$(a+d)\mu^2+(-a-3b+2d)\mu+b+2c+d<0$$

The left-hand side of such inequality is an upward bending parabola owing to (5). Therefore, it crosses the horizontal axis twice and takes negative values, if and only if the discriminant (13) is positive.

Remark 2. Consider all efficient or maximum-mean portfolios above the minimum-variance point *T* in $(s;m)$ space. According to Proposition 3, if the discriminant (13) is positive, such portfolios form a line that has both a local maximum and a local minimum. Figure 4, a case in point, is based on the annual nonoverlapping **total** returns of four major equity classes over the time period from 1972 to 2005. As explained in Section 2, the data set includes US stocks (S&P 500 index), international stocks (MSCI EAFE index), real estate securities (FTSE NAREIT Equity REITS index), and commodity-linked securities (S&P GSCI index). However, if Table 2 were updated, most strategic asset allocations would include a negative weight.

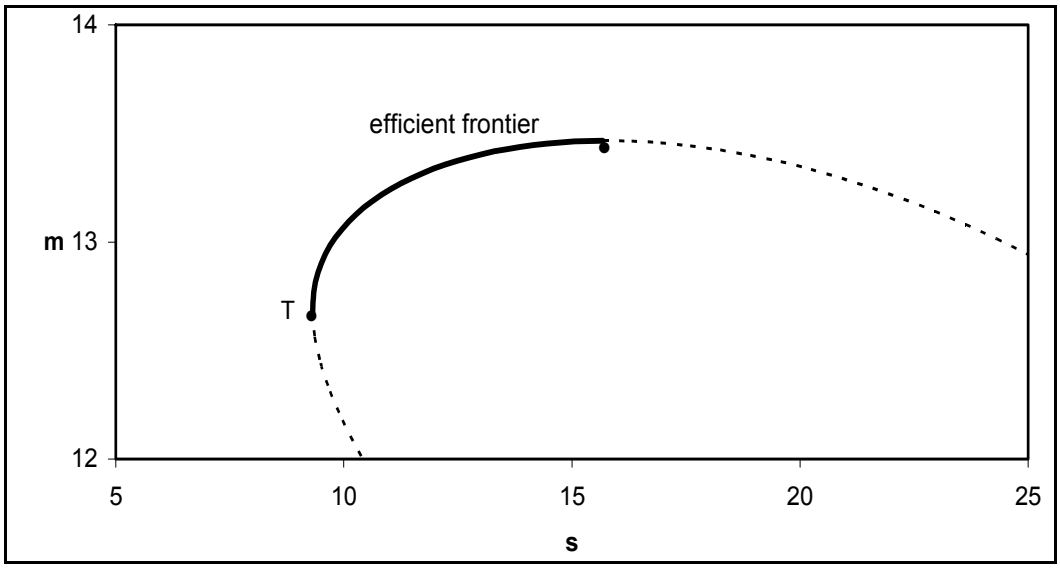

Figure 4 – Another alternative efficient frontier

6. Discussion and conclusions

A problem of passive management has been taken into consideration. Capital is invested into a portfolio that is **rebalanced annually** to restore the percent weights of its **strategic asset allocation**. Its annual total returns r are time incorrelated and stationary; commissions, fees, and personal taxes are disregarded. Two sets of historical data have been considered; for the sake of simplicity, **population moments** have been supposed to be the same as historical moments.

Mean-variance optimisation has been performed, using the **population moments** of **simple returns** as inputs and obtaining all **efficient portfolios**, i.e. all optimal sets of weights, as outputs. Therefore, the hyperbola represented by Equation (4) has been retrieved, whereby the mean return of each efficient portfolio is matched by the smallest possible standard deviation.

Next, the **simple returns** of each (efficient) portfolio have been supposed to be **lognormally** distributed. Reference has been made to the one-to-one mapping (8)-(9) between the **population moments** of **simple** and **logarithmic returns**. Expanding on Elton and Gruber (1974) and Merton (1972), it has been shown that such a mapping turns a **minimum-variance set** based on **simple returns** into a **minimum-variance set** based on **logarithmic returns**. Although the two minimum-variance points don't coincide, both sets have a similar shape. Moreover, inefficient portfolios based on simple returns cannot map into efficient portfolios based on logarithmic returns, whereas efficient portfolios based on simple returns can map into inefficient portfolios based on logarithmic returns according to two alternative qualitative patterns. According to historical data, efficient portfolios display simple and logarithmic returns with similar ratios $\frac{\mu}{\sigma}$ and $\frac{m}{s}$ between population mean and standard deviation. Needless to say, our theoretical results don't apply solely to passive management, as they have a more general validity within portfolio theory.

Now suppose that a forward looking analysis is performed over a future time period, which is n years long. As portfolio returns are **lognormal** by assumption, Equation (1) can be rewritten as

$$\text{final accumulation} = \text{initial capital} \cdot e^{\left(m \pm z\frac{s}{\sqrt{n}}\right)n} \tag{16}$$

where z is a quantile of the standard normal distribution in line with an appropriate confidence level and

$$e^{m \pm z\frac{s}{\sqrt{n}}} - 1$$

is the corresponding confidence interval of the sample **geometric** mean. If emphasis is placed on the **downside risk**, the **shortfall constraint approach** may be taken (Leibowitz and Henriksson, 1989) so as to choose the most appropriate efficient portfolio. Accordingly, one has to pinpoint the portfolio **threshold return** $\ln(1+\tilde{r})$ *per annum*, derive the linear shortfall constraint

$$\ln(1+\tilde{r}) = m - z\frac{s}{\sqrt{n}} \quad \text{i.e.} \quad z = \frac{m - \ln(1+\tilde{r})}{\frac{s}{\sqrt{n}}} \tag{17}$$

then take the alternative efficient frontier into account, and eventually obtain the **shortfall probability**, and hence the confidence level, that is attached to each efficient portfolio. Therefore, each efficient portfolio is such that the portfolio **threshold return** $n\ln(1+\tilde{r})$ is exceeded after n years with the due **shortfall probability**, i.e. the due confidence level. Remarkably, two novelties mark our setting: allowance has been made for the compounding of portfolio returns over a n year period, portfolio returns being **lognormal** rather than **normal**. Such an assumption might be an acceptable stretch; for instance, it is met by the annual logarithmic returns on stocks owing to the well known aggregational lognormality (Chakraborti *et al.*, 2011, p. 6). Equation (16) implies that the mean logarithmic return can be turned into a forecast accumulation, whereas the **threshold return** can be turned into a **threshold** accumulation that has the same confidence level. In contrast with Leibowitz and Henriksson (1989), different time periods n can be considered by using only annual portfolio moments; Equation (17) entails that the more distant the time horizon, the larger is z and the smaller is the **shortfall probability**.

However, the use of s in Equation (16) may be too conservative, as stock prices are **mean reverting** in a statistical sense. According to the Royal Swedish Academy of Sciences (2013), the empirical researchs by Eugene Fama, Lars Peter Hansen, and Robert Shiller, 2013 Nobel laureates, have shown that stock prices are unpredictable over the short term, i.e. the next year, and predictable over the medium term, i.e. the next 3-5 years, owing to both rational and emotional causes. Unfortunately, mean reversion doesn't imply that stock returns are independent (and identically distributed). Allowing for mean reversion goes beyond the scope of this paper; moreover, it is unclear whether a more complicated optimisation model might be both analytically tractable and operationally useful. An appealing and alternative heuristic

approach calls for the use of Equation (16) provided that an educated adjustment is made, putting up slightly the target shortfall probability, i.e. lowering slightly the target confidence level.

In our opinion, the alternative efficient frontier based on logarithmic returns may be useful both in teaching and business practice, as it provides a more solid foundation for choosing an appropriate strategic asset allocation. Moreover, an analytically tractable and operationally useful alternative would be hard to find. In the light of Hakansson (1971), it is unclear whether, in the general case, efficient portfolios based on simple returns may also turn into efficient portfolios based on logarithmic returns. Finally, the linear-quadratic model represented by Equations (2)-(3) is susceptible of extensions, which are left to future research. On the one hand, additional constraints, e.g. nonnegative weights, can be considered so as to obtain a linear quadratic model that is more in line with passive management; on the other one, a sensitivity analysis can be performed so as to determine the most critical population moments.

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Sommario

Si considera l'uso della frontiera efficiente nell'ambito di un problema di gestione passiva. Un portafoglio aggregato viene ribilanciato una volta all'anno in modo da ripristinare i pesi percentuali della sua allocazione strategica; i suoi rendimenti totali annui sono per ipotesi indipendenti e lognormalmente distribuiti. Estendendo precedenti risultati teorici, si mostra come un insieme a minima varianza associato ai rendimenti semplici si trasformi in un insieme a minima varianza associato ai rendimenti logaritmici. Secondo i risultati teorici ottenuti, che hanno una validità generale, portafogli inefficienti in termini di rendimenti semplici non possono trasformarsi in portafogli efficienti in termini di rendimenti logaritmici, mentre portafogli efficienti in termini di rendimenti semplici possono pure trasformarsi in portafogli inefficienti in termini di rendimenti logaritmici. Nel secondo caso, sono possibili 2 differenti andamenti qualitativi, entrambi illustrati avvalendosi di dati storici. Inoltre, si estende l'approccio con vincolo di ammanco al caso di portafoglio avente rendimenti lognormali. Ogni rendimento soglia può essere trasformato in un montante soglia avente la stessa probabilità di ammanco; *coeteris paribus*, più distante è l'orizzonte temporale, minore è la probabilità di ammanco. Poiché il procedimento analitico è agevole, può risultare utile nella pratica professionale, specialmente per gli analisti finanziari e gli investitori istituzionali.

Abstract

An efficient frontier model is derived within a problem of passive management. An aggregate portfolio is rebalanced annually to restore the percent weights of its strategic asset allocation; its annual total returns are assumed to be independent and lognormally distributed. Expanding on previous theoretical results, it is shown how a minimum-variance set based on simple returns turns into a minimum-variance set based on logarithmic returns. According to the attendant theoretical results, which have a general validity, inefficient portfolios based on simple returns cannot turn into efficient portfolios based on logarithmic returns, whereas efficient portfolios based on simple returns can also turn into inefficient portfolios based on logarithmic returns. In the latter instance, there can be two different qualitative patterns, both of which are portrayed by using historical data. Moreover, the shortfall constraint approach is extended to the case of lognormal portfolio returns. Each threshold return can be turned into a threshold accumulation that has the same shortfall probability; *coeteris paribus*, the more distant the time horizon, the smaller the shortfall probability. As our procedure is analytically tractable, it might be operationally useful, especially to financial advisors and institutional investors.

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