FUZZY NORMS, DEFAULT REASONING AND EQUILIBRIUM SELECTION IN GAMES UNDER UNFORESEEN CONTINGENCIES AND INCOMPLETE KNOWLEDGE. [1 2]

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1. Introduction and motivation

This paper focuses on the role that norms play in the selection of equilibrium points seen as social conventions under unforeseen contingencies – that is in the emergence of regularities of behaviour which are self-enforcing and effectively adhered to by limited rational agents due to their self-policing incentives.

We are not concerned here with the explanation that norms induce equilibrium points in games because of their structuring effects on payoffs and incentives. Equilibrium points are seen here as pre-existing possible strategies' combinations of a given game situation. Moreover norms are primarily seen as sets of logically consistent normative statements prescribing a unique action to each player in any game situation of a given class of games. Hence, these statements demand players to follow a given equilibrium behaviour within the existing equilibrium set of the given game situation. Their effectiveness has not to be seen in that they provide enforcement of some set of actions by means of external incentives or sanctions, but only in their functioning as cognitive devices inducing the appropriate system of expectations on equilibrium behaviours. Given the appropriate system of expectations, players have endogenous incentives to follow the prescribed behaviour, as long as they expect the corresponding actions are also executed by the other players.

To explain, let split the reasoning in steps. First, any norm is meant as a set of normative statements saying that a given rule of behaviour has to be conformed to. Second, strategies complying with norms are recognised to be reciprocal best responses, components of an equilibrium point. Third, playing the correspondent equilibrium strategies is supported by the belief that (for whichever reason) the norm has got general acceptance. If players have got these beliefs, then they will expect that all will act accordingly, and given these mutual expectations, *then* they will find the endogenous incentives to follow the recommended strategies because of their equilibrium properties.

But notice that there are many equilibrium points in any non trivial non cooperative game. Mixed interests coordination games or the typical iterated Prisoners' Dilemma, where Folk theorem does applies, can be thought as the natural classes of game situations to which norms refer. Given many equilibriums, hence, there are also many norms in the sense defined. We may also think that there are even more norms, in the sense of sets of statements prescribing individual actions to the players if some of them could also recommend to endorse behaviours that do not satisfy the equilibrium property. Even if the possibility of such utterances cannot be excluded in principle, we disregard them, as they do not satisfy our definition of norms as a set of sentences prescribing to follow pre-existing equilibrium strategies in a game. We may term such further norms as "prima-facie-norms" – to say. The main multiplicity problem however remains: given multiplicity of equilibriums - such that the expectation of each equilibrium supports the decision of following a different norm in turn - which norm should be chosen or followed? We assume that an equilibrium gains its justificatory power in terms of individual rational choice since shared knowledge that the equilibrium itself will be adopted as the solution of a game has gained general acceptance. Hence, multiple equilibriums are situations such that, even if we know that *an* equilibrium will be adopted, nevertheless there is not implication about *which* equilibrium will be adopted and as a consequence we do no have any reason to act upon any of the possible equilibrium. Hence no incentive-based reason to follow any norm is effective. Unless we know which equilibrium is to be adopted in order to solve the game, we cannot say to have the system of belief concerning the equilibrium behaviour grounding a given norm in terms of the incentive-based reasons to act upon it.

What we have to look for relates to the explanation that equilibrium points are selected because they are *focal points* or exhibit what is usually called *salience* (Shelling 1960, Lewis 1968). In those approach where *focal points* are not seen as primitive brute psychological facts attached to behaviours, but as the result of existing social norms (Kreps 1990), norms work as determinant of the state of knowledge that makes predictable equilibriums as outcomes of the game, providing in turn endogenous incentive to follow the relevant norm. There is however a sort of circle: equilibrium points are selected because of their *focality*, which in turn is due to social norms that give "salience" to that equilibriums. But on the other hand this can be so only because some of the multiple norms are *focal points* on their own, able to co-ordinate expectations in the relevant class of games, which needs no less explanation than the salience of equilibrium points as such.

Thus more hypothesis are needed than the simple one that norms are set of statement demanding to act according to a rule of behaviour which coincides to an equilibrium point of the game. This is not at all sufficient in order to predict that an equilibrium will emerge from strategic interaction in a game. It must be assumed moreover that beyond the normative meaning ("do that") and the coincidence with an equilibrium of the game ("if all the player follow the rule *then* they have the incentives to do that because it coincides with an equilibrium point"), players do also entertain mutual expectations that they all will follow the rule, i.e. there must be shared knowledge that the norm is followed if it must be true that the norm itself will induce the appropriate set of expectations and incentives. It is only *because* it is shared knowledge that (i) all the players follow the norm and *hence* that (ii) in the given game the strategies recommended by the norm will be employed by every players, *that then* they have the strategic reason to adopt the recommended strategy *because* – at last - it is the best response to the expected actions chosen by others.

As just mentioned, there is a risk of circularity in this format of reasoning: *because* there is a system of expectation such that everybody knows that the norm will be complied with in general (for a given class of game), *than* there is the system of reciprocal iterated expectations that everybody will act accordingly to a particular equilibrium point in the game at hand. As a consequence, every players have the appropriate game theoretical reasons to play the given equilibrium solution (the "best response" reasoning). *Then* it can be inferred that it is common knowledge that this equilibrium point will be the solution of the game. This seems to turn the conclusion on its own premise round. In order to escape the vicious circularity, it must to be kept in mind that the inference to common knowledge of the game's solution presupposes some kind of shared knowledge or expectations about norm-following behaviour in the given game (and in the game class in general), other than the knowledge of the best response property of any player strategy given the expectation upon the other players choice.

The relevant question to ask then is: where does the basic shared knowledge about norm-following behaviour come from? It must exist an independent process of reasoning carrying us to believe that everybody will follow the norm and the prescribed behaviour within the given game, in order to generate the system of expectations about the strategies adopted in the game, which rationally justifies a player in adopting his equilibrium strategy *because* of its best response property. We want to counter this problem by investigating how players *arrive* to believing that in a *given game* all the players will conform to a general norm, *before* they begin calculating that, *because* it is known that players will conform to the norm, *then* they play the given equilibrium. To simplify, let we introduce the *ex ante/ex post* distinction about players' reasoning processes: it is *ex post* reasoning to say that, *because* it is common knowledge that a given equilibrium is the solution of the game, *then* players do have the incentive to conform to it. On the contrary it is an *ex ante* reasoning that carries us from the knowledge of a pre-existing norm, having prescriptive meaning and being generally accepted as the solution concept for a given game class, to the conclusion that this norm is in fact accepted as the solution theory also in the *given* game we are involved in, so that the *ex post* common knowledge-based reasoning may then emerge. By the equilibrium selection process we mean the *ex ante* reasoning that bring us to a state of shared knowledge that a given equilibrium will be the solution of a given game since the game situation confront player with the choice amongst many equilibrium strategies.

Equilibrium selection is a wide topic in game theory (Harsanyi 1975, Harsanyi-Selten 1988, Binmore 1987/88, Fudenberg-Levine 1998, Samuelson 1997). Harsanyi and Selten put forward the first theory of equilibrium selection, based on a modelling of the outguessing regress idea that the players reciprocally simulate the reasoning process one of the other, by repeatedly assigning to the counterpart their own previous reasoning steps – an approach that Binmore calls "eduction" and that we want to pursue in this work. However they build up their theory under too strong epistemic assumptions - so strong that not even their equilibrium selection model can satisfy them. Common knowledge of the prior– something like the point of view of an external rational observer about basic uncertainty that all the players know and dictates them the only appropriate way to formulate their ignorance about initial equilibrium choices – is incompatible to the assumption of a common prior as starting point of the equilibrium selection process, i.e. a shared way to formulate uncertainty about the possible initial equilibrium choices in the game (Sacconi 1995b)[5].

On the other hand we are looking for a solution to the equilibrium selection problem not usually addressed to, as it ranges over games where equilibrium points have to be selected by players facing incomplete knowledge of the possible states of the world and consequently of the game they are involved in. Multiplicity occurs in this cotext because of the ambiguity concerning the game players are going to play: This can be understood as if they were able to predict the single equilibrium by which they will solve a game if there were no ambiguity about the game they are going to player. But, nevertheless multiplicity come back due to ambiguity about the game they are going to play occasioned by unforeseen contingencies. This is not the same as extending existing approaches to games under incomplete information (Harsanyi 1967/68). Our approach is modelling *how* players come to the conclusion that a given game *falls* into the domain of a *pre-existing* general norm, while we must nevertheless *assume* that, as far as a game where this norm does apply is concerned, it is common knowledge that the norm is conformed to by all the participants. The point to be understood is how players *come to recognise* that a given game situation is an exemplar of a general abstract norm so that they can reasonably believe that in the game to be played they must follow that norm and everybody else must also follow the same norm.

In a sense we do not eliminate the presumption that there exists some “common knowledge” in the world about how players will act given certain norms and forms of interaction. This can be retained. But we want to minimise its impact. In fact we break the just mentioned line of the apparently circular deduction where it says that *because* we know that a general norm is followed in some game class, *then* everybody knows that in the game they are playing (which belongs to the class) the norm will be followed and this implies that a given equilibrium is expected to be used in solving that particular game. This chain of reasoning is broken where it is said that *because* there is a norm and it is shared knowledge that it is followed in a certain class of game situations, *then* it is also shared knowledge that it will be followed in *our* game. This inference cannot be worked out as a matter of pure classical logic nor it can be taken for granted as a primitive psychological fact. It must be explained by an explicit modelling of the limited reasoning process performed by the players when they face game situations and reason as to recognise whether these game can be understood according to the pattern of a solution theory devised for a general abstract game form. It is by this reasoning that we can explain where the required “common knowledge” (or something less than so, that can perform the same role - call it “shared knowledge”[6]) does come from the *concrete* game situations the players face.

We plan to show that this reasoning employs fuzzy pattern recognition and default logic. These two building block of the equilibrium selection reasoning process can be seen as a new account of how deliberative rationality, together with general abstract norms, works in aiding equilibrium selection in games by subsuming new games under the domain of norms when this is not a matter of logical consequence, but a matter of genuine default extensions of the domain of the existing general norms. Of course all this makes sense exactly in the contexts where norms do make sense: games played under incomplete knowledge about unforeseen states of the world (that is to say a typical domain if bounded rationality).

The paper proceeds as follows. First (sec.2), we define two versions of a basic game, such that in each of them we assume that players know that a given solution (norm) is shared knowledge (the Nash product for the "demand game" version and the typical non cooperative solution for the "enlarged DP" version – both coinciding to equilibrium points of the two versions of the basic game). Then (sects. 3 and 4), the basic incomplete knowledge situation is introduced by assuming that a move of Nature selects states of world where one of the two versions of the basic game will be played. However some of the states of the world that nature may choose are unforeseen. Thus, when nature has made its choice, players will face states that they *ex post* know to occurs, but that are vaguely described in terms of their *ex ante* knowledge about the rules of the game. Unforeseen contingencies are then modelled as states that can belong to events defined in terms of the *ex ante* base of knowledge only through fuzzy membership functions (Sacconi 2000). Players must decide which strategy to play under this characteristic lack of clear information, i.e. under vagueness on the game they are going to play. Their information is resumed by fuzzy measures of membership that induce possibility distributions over states, but does not allow them a shared knowledge of the norm (solution), which has to be conformed to in the game to be played in the given state. Consequently there is no basis at this stage to infer that everybody knows that a given equilibrium will be played and to conclude that a player must use his best response belonging to a given equilibrium solution.

Next, default reasoning enters the scene. We first (sec. 5) introduce default logic according to Reiter (1980) and we attempt a rough approach to the default reasoning players perform in order to assign uniquely solution concepts (norms) to unforeseen state of the world (sec.6). However, this rough approach is instructive but inadequate to the scope of equilibrium selection. Hence in sec.7 a proper selection process is introduced, as it is based on the reformulation of default logic in terms of possibility theory (sec. 7.1) given - after Zadeh (1978)- by Dubois, Prade and Benferhat (Bubois-Prade 1995, 1996, Bubois-Prade-Benferhat 1992). At the first step of the recursive reasoning process each player's first hypothetical choice, given the basic vague knowledge about the game they are going to play, is calculated in terms of a fuzzy expected utility function (Yager 1979) - where fuzzy utility is combined with a possibility measure on single unforeseen states (sec.7.2).

At the second step (sec. 7.3), each player guesses the reasoning step carried out by the counterpart. Maybe fallibly, each player attributes by default to the other player his own scheme or reasoning, because it appears to be the most "normal course of reasoning" to him - since he does not know about any falsification of his own scheme of reasoning till then. This is provided encoding by the formulae of a formal language the knowledge base that players have on the game and their default rules of inference - enunciates like "normally players own such information" or "normally a player who owns such information plays strategy such and such in a game like this". On these formulae we are able to induce a possibility ordering. The ordering will represent constraints on what the players believe as the "normal course of facts", which are imposed by the default rules that extends the players' base of knowledge.

Then, at the third step (sec. 7.4), we may calculate each player second hypothetical choice given the reconstruction of the other player symmetrical reasoning and the ensuing new possibility that the player chooses any action in each game, given any state. In order to accomplish this calculation the player must combine two pieces of vague knowledge: the first order vague information expressed by the ex post possibility distribution over unforeseen states of the world and the second order vague information resumed by the possibility of each action in each game given any state, which is derived by a default inference simulating the counterpart's reasoning. This concludes to a new best response for each of them in fuzzy expected utility terms. Iterating the procedure will not change the set of predicted choices. So that we can conclude (sec.7.5) that the default-possibilistic reasoning will stabilize in a couple of strategy choices that constitutes one of the two basic games equilibrium points. We end up by suggesting that the resulting equilibrium will be supported by what in default logic is called an "extension" of a given theory - which is usually characterised as a fixed point (Reiter 1980)- obtained by the iterate application onto itself of the set of accepted defaults, without introducing any contradiction. A remarkable feature of our result is that the equilibrium reached depend in essential way on non-monotonicity of default logic, and hence it allows for mistakes and revisions in the equilibrium selection process, which seems to pertain to the very nature of bounded rationality.

2. Two games of reference

Consider the typical demand game. 100 dollars are to be divided in two shares according to demands made by two players. If shares sum up to 100 or less, the players gain their demands, but if demanded shares sum up to more than 100, they do not get anything. Players participate in the game announcing a proposals of sharing that they accept and that they think can be compatible with the proposal of sharing the other player is simultaneously announcing. In fact, they announce demands and offers at the same time and only once. They can also refuse to demand and offer anything and this will end the game, notwithstanding the other player proposal, even if he would be ready to concede all the sum to the first player. If one or both refuse to agree, the result is the *status quo* (zero dollars). Thus players need to guess demands that may be acceptable to the counterpart, under the condition that only couples of demands that sum up to no more than 100 are feasible.

Let assume (this is where the announced assumption of background shared knowledge comes in) that players know that in games like this there is a unique rational solution, that is maximising Nash bargaining function (N.b.f.) which, under the appropriate utility representation, coincides with the 50-50 sharing rule. We simply assume that it is shared knowledge that in this game players will follow the Nash bargaining solution because they accepts the bargaining axioms given by John Nash (or the equivalent Harsanyi-Zeuthen's axiomatisation). This is an abstract norm generally accepted for solving games like this and it is shared knowledge that it is so. It also requires one more condition: games like the demand game are *cooperative bargaining game*, that is any demand and offer made by a player is binding for him and no player can defect from them, which means that he has to respect and to comply with any agreement once it has been reached. When two compatible proposals are announced the resulting pair of demands and offers determines an agreement that will be necessarily executed. The player cannot defect from that, cannot refuse of conceding the share offered and cannot expropriate the full sum by cheating the other party in the agreement. This amount to say that within the demand game it is shared knowledge that the game is *cooperative*, which implies that one more rule - *pacta sunt servanda* – does apply to it and it is shared knowledge to be binding. The reasons for it is known that this is an effective rule of the game can be many: it may be true that no players are in the physical capacity to defect from announced agreements, or players dislike to defect, in the sense that the payoffs they attach to defection are zero. The game could be embedded in an un-modelised even more large repeated game, in which respecting pacts is a self-enforcing convention self-polished by the threat to be punished or loosing reputation in subsequent iterations. We do not ask why this is so, and simply assume that as far as the players understand the demand game so that it is shared knowledge that N.b.f. is the accepted solution, then it must also be understood that players cannot successfully defect.

Next, let assume that players also know that there are somewhere games where this is not the case. As far as the opportunity of sharing 100 dollars by an agreement is concerned, these further game are completely identical to the demand game. However those games are termed *non-co-operative* and are game situations that do not satisfy the requisite that all players accept the rule that *pacta sunt servanda*.

See below two matrix games quite similar one to the other, the difference between the two being that in the second game players have the capacity to defect and this is consequential to their payoffs, whereas in the first – which is the demand game - this capacity is nil. This inability for defection is captured in G1 by the non-co-operating strategy D, which identifies with keeping to status quo, without getting any advantage from not co-operating. Remaining strategies have to be read as follows: any player may propose three possible sharing of 100 dollars (33-66; 50-50; 66-33, the first of the two numbers being the proponent's payoff). If it comes out that the proposals are compatible, in the sense of not exceeding the total sum of 100, they are automatically accepted and translated into payoffs. If proposals are incompatible the result is nil for every player (*status quo*) They can also intentionally refusing to agree (keeping the *status quo*), by playing D. This decision has not positive effect on the player's utility, whereas it may frustrate the counterpart willing to take advantage by participating in the game.

The players' utility functions are taken to be identical to linear transformations of the monetary payoffs mapping money onto the real line [0,1], so that 100 corresponds in utility to 1, 66 corresponds to 0.66, 50 to 0.5, 33 to 0.33, 0 to 0. The reason for adopting these particular utility functions will become clear later. The game is given a matrix form as it is meant to stress that the pairs of mutually adapting demands summing up to 100 are Nash equilibria of a simultaneous moves non-cooperative tacit coordination phase (in fact there is no explicit bargaining in the game as the players have only one round to make proposals). Thus the problem of choosing one equilibria amongst the possible three is represented as non-cooperative in the tacit coordination (bargaining) phase.

Game G1:

	33	**50**	**66**	**D**
33	0,33 / 0.33	0.5 / 0.5	0.33 / 0.66	0 / 0
50	0.5 / 0.33	0.5 / 0.50	0 / 0	0 / 0
66	0.66 / 0.33	0 / 0	0 / 0	0 / 0
D	0 / 0	0 / 0	0 / 0	0 / 0

Any pair of demands, satisfying the 100 constraint, can be enacted as an agreement in the un-modelled implementation phase anyway, and this reflects the underlying cooperative assumption. However the underlying co-operative nature of the game is also pointed out by the fact that D (defection) is ineffective and coincides with simply the status quo. Moreover it is assumed that, according to cooperative bargaining theory, the players will accept 50-50 as the unique rational solution. In fact the matrix game can be seen as a rough approximation to the Nash bargaining game, where the status quo is zero for both and the payoffs space is the convex (compact) utility representation of any possible proposal of sharing that sums up to 100 or less than so, including also any linear combination mixing up any two or more pure proposals of sharing.

The main assumption in game G1 is that players accept the cooperative theory of bargaining (a set or normative statements), this is shared knowledge and then they are expected to adopt the corresponding equilibrium - the one maximising the product $\Pi_{i=1,2}(u_i - d_i))$, where in this cotext the status quo d_i to any player I is 0, that is (50,50) = (0.5, 0.5). This builds up the required system of mutual expectations that in the game gives reason to follow the solution. Thus, we say that games like G1 belongs to the domain of a general abstract norm: the solution theory σ1, i.e. N.b.f.

Game G2:

	33	**50**	**66**	**D**
33	0,33 ; 0.33	0.5 ; 0.5	0.33 ; 0.66	0 ; 1
50	0.5 ; 0.33	0.5 ; 0.50	0 ; 0	0 ; 1
66	0.66 ; 0.33	0 ; 0	0 ; 0	0 ; 1
D	1 ; 0	1 ; 0	1 ; 0	0.21 ; 0.21

Next consider the second game G2 made out of a slight payoff change of the demand game G1. In G2 the defection strategy D has an effective role to play, as the players are now able to free ride the counterpart by refusing of co-operating and at the same time by appropriating of the entire sum if the counterpart decide to make a proposal of agreement. This can be read as cheating the other player and reaping the entire sum after the other player has made some proposal. For example let the second player to make any proposal. What he immediately sees is the first player that refuses of co-operating by leaving the bargaining table and overthrowing it. Next, however the first player has a more accurate look at the picture and he understands that the entire sum to be shared has been stolen. According to the rule of the demand game that sum should have been wasted for both the players. On the contrary in game G2 the defecting player may reap the entire sum by taking unilaterally advantage of the other player disposition to agree. There is no assumption of any underlying co-operative structure to this game. Moreover G2 is meant to represent that no agreement has any chance to be self-enforced as the defection strategy is dominant in the game (which is in fact an enlarged PD game). The explicit representation through the D strategy of the ability to frustrate any cooperative endeavour and gaining advantage by this conduct shows that the underlying game situation does not belong anymore to the domain of cooperative game theory. *Pacta* are not seen by the players as to be *servanda*, and there is shared knowledge that the rules of the game does not imply that any agreement will be enforced by some un-represented mechanism on the back of the situation. Moreover there is no shared knowledge that N.b.f. is the accepted norm in this game. On the contrary there is a quite different norm that is accepted by all the players and which is shared knowledge to be accepted amongst them: the solution theory σ2, i.e. "if there is an outcome in dominant strategies, it can be expected to be the solution of the game". Here this is enough to select the unique solution (D,D)[7].

3. Unforeseen states of the world: vagueness on the appropriate solution concept

The foregoing section was only preparatory to our problem. Consider the following game situation. Two player P1 and P2 are called to play a game under incomplete knowledge, which means that they can be submitted to choices by Nature that selects states of the world they were not yet able to imagine before the start of the game. We assume that they know the two abstract class of games - the demand game and the enlarged DP game, depicted above. They have the necessary background in game theory, thus they also know the accepted theories of solution to be applied to the two prototypal game situations, i.e. the two solution theories σ1 and σ2 that, according to our point of view, are two established social norms. They of course have shared knowledge that if the game Nature pushes them to play were G1 then it would be obvious to play according to the norm σ1 (N.b.f.), but if Nature were to put them into a G2 game then it would become obvious that the norm σ2 has to be adhered to. This is to say that P1 and P2 have in the back of their minds the representation of a set of possible states of the world which all and only match with one of two possible events - having to play G1 or G2 - and they expect to be confronted by nature's choices that will put them alternatively under states where alternatively G1 or G2 are to be played.

The game start with a Nature's move. Nature may select one out of two states w_1 and w_2, in each of them being true that either G1 or G2 (but not both) will be played. However Nature may also select (or, to say more exactly, let the players to discover) a set Ω of unforeseen (by the players) states of the world, which are states that do not conform to the properties and predicates in use into P1's and P2's language, by which they in general describe games. Due to their unforeseen characteristics, such states do not match exactly the foreseen events the players are prepared to learn. This is to say that in each $\omega_i \in \Omega$ it is *nor* true *nor* false that G1 or G2 is to be played. Notice that this is not the usual game under uncertainty: it is not the case that players have information partitions defined on the possible states' set, which are not refined enough to discriminate exactly the occurrence of states where it is true that the game G1 is to be played from the occurrence of states where on the contrary it is true that game G2 is to be played. After Nature's choice they perfectly discriminate amongst the states and there is not uncertainty about which state has been reached. The situation is meant on the contrary to represent ambiguity or vagueness the players face as long as *any* state is scrutinised about whether in that particular state the game G1 or G2 is to be played. Vagueness is the consequence of they not having *ex ante* knowledge of the entire set of states that *ex post* Nature reveals to be possible, which implies that these are unforeseen states. Thus vagueness is the consequence of the players not having the conceptual and linguistic tools to clearly describe in any detail each $\omega_i \in \Omega$ or, in other words, the consequence of each $\omega_i \in \Omega$ being unclear about the game to be played in it. If it were *ex post* clearly specified whether in any state $\omega_i \in \Omega$ the game G1 or G2 is to be played, then the players would have been able to foresee these states at least as states of the world that *ex ante* they knew to be possible.

According to one of us (Sacconi 2000, 2001) however unforeseen states of the world can be analysed in terms of fuzzy events, which are the fuzzy sub-sets of the reference set Ω. A general norm, construed by the theory as the solution for a given abstract class of games, has a domain of application (a set) which is defined by the membership functions of any state into the set representing that domain. Foreseen states have crisp membership function (we were already able to say whether these states, if occurred, would have instantiated a case of the norms). Unforeseen states on the contrary have fuzzy (graded) membership functions taking their value into the real line [0,1] and expressing the degree at which the *ex post* revealed states can be recognised as belonging to the domain of a given norm. Vagueness has not to do with the normative content of the norms, or solution theories - the prescription to use the strategy D or the strategy 50. It concerns the descriptive premises for those norms, i.e. for example whether the game can be described as *cooperative* or *non cooperative*, whether the descriptive condition, necessary in order to say that *pacta* are seen as *servanda,* are satisfied.[8] We capture this point by introducing vagueness in the *ex post* description of the "defect" strategy in any ω_i state of the world. Being the D strategy describable like in G2 means that the game is not cooperative and that it does not belong to the domain of the solution theory $\sigma 1$. Conditional on the description of strategy D, we could say whether the event G1 or the event G2 occurs. However this is a matter of vagueness under unforeseen states of the world.

So far, we had two possible norms each prescribing a unique equilibrium point of the game within which it is the accepted solution theory. However in the game under incomplete knowledge and unforeseen states players cannot say at first glance which equilibrium point will take place, because they cannot say which solution theory has to be employed. We want to see whether general and abstract norms provide at least the starting point for a norm-based equilibrium selection reasoning procedure, which at end shall be able to decide the equilibrium point, belonging either to game G1 or game G2, that will be played as the unique solution of the game. As expected, the answer will prove to be dependent on Nature's choice of the state of the world. This doesn't introduce any uncertainty nor make room for probabilistic nature of the equilibrium selection procedure. Nature makes first its choice and then players learn it without any uncertainty before they reach their own move in the game. Were Nature not selecting unforeseen states of the world, players P1 and P2 would learn without any uncertainty whether they are going to play G1 or G2, and would solve each game according to its accepted solution theory. Thus, we have only to investigate what happens when each $\omega_i \in \Omega$ is chosen by Nature.

4. Solutions' domains as fuzzy sets

Let start by assuming that Nature has two consecutive moves at the beginning of the game. First it chooses amongst w_1, w_2 or Ω. In w_1 and w_2 it is true that games G1 or G2 are played respectively. We disregard these branches of the game tree because they have a completely obvious solution each. Next Nature has its second move amongst states in Ω.

Take Ω to be the set of all the possible, alternative, complete descriptions of the interaction situations between players P1 and P2, which are workable out by affirmation or negation of any concrete property that may influence the structure of their interaction (expressible by predicate in the language they develop to describe Ω). Assume that any $\omega_i \in \Omega$ typically specifies the characteristics of the players, the principles of ethics they follow, the set of options open to them, the environment in which they are inserted and every other property that makes it possible to say whether the norm *pacta sunt servanda* holds or doesn't hold. Thus, Ω is the "universe of discourse" containing all the possible concrete game situations the payers may be involved in. Remember however that some of these concrete characteristics are genuinely "unexpected" and are not univocally expressible by means of terms belonging to the *ex ante* language, which was in use when the description of the two games G1 and G2 was in order. In particular under any state $\omega_i \in \Omega$ some of characteristics are not traceable back without any ambiguity to the conditions established as necessary to say whether the game form G2 or G1 are the appropriate abstraction from the concrete game situations under consideration. Thus, it is not clear whether having such characteristics or their negation implies that game G1 or game G2 will be played. [9]

Then, let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ be the set of *ex ante* unforeseen states that Nature may select at its second move. Hence define two fuzzy events

$$\underline{G}1=\{\omega_i, \mu_{\underline{G}1}(\omega_i) \mid \forall \omega_i \in \Omega\}$$

$$\underline{G}2= \{\omega_i, \mu_{\underline{G}2}(\omega_i) \mid \forall \omega_i \in \Omega\}$$

For example let them to be

$$\underline{G}1 = \{(\omega_1, 0.9), (\omega_2, 0.5), (\omega_3, 0.2), (\omega_4, 0.1)\}$$

$$\underline{G}2 = \{(\omega_1, 0.2), (\omega_2, 0.4), (\omega_3, 0.5), (\omega_4, 0.8)\}$$

The fuzzy membership functions $\mu_{\underline{G}1}$ and $\mu_{\underline{G}2}$ represent the vagueness associated to the occurrence of each state concerning whether the game to be played is G1 or G2. As $\underline{G}1$ and $\underline{G}2$ are sets of states, defined by associating to each unforeseen state ω_i its membership function to the relevant set, they may be understood as *fuzzy events* that occur when unforeseen states are selected. These sets may also be understood as the vague domains of application of the two solution theories $\sigma 1$ and $\sigma 2$ respectively (for example $\sigma 1$ does apply to ω_1 at degree 0.9).

Notice that you cannot interpret the fuzzy distribution over states as a probability distribution over states, nor to sum up the fuzzy degrees in order to compute the total probability of the fuzzy event. On the contrary you can consider any fuzzy event as a flexible constraint over the *possibility* of states, according to the expression

$$\pi_{X is \underline{G}j}(\omega_i)= \mu_{\underline{G}j}(\omega_i), \qquad i=1,2,3,4; \quad j=1,2$$

Let X to be a variable ranging over the set of states Ω constrained by the fuzzy information that "X is $\underline{G}_i$", that is the game played in the state selected out of Ω is $\underline{G}_j$, j=1,2, according to the relevant membership function. To say that X is a variable ranging over the universe of discourse Ω means that X takes its values from the set of concrete descriptions of the possible interaction situations between P1 and P2, which in fact is what any state in the set Ω describes. The players' information constrains the possibility that X takes any value in Ω, i.e. the admitted values of X are confined to those that are compatible with the information that any given ω_i belongs to G1 or G2. Given the unexpected nature of these states, however the variable X is constrained by fuzzy information about the identification of any interaction situation with the game form G1 (G2 respectively). Thus the possibility that any state ω_i, given the fuzzy information that the game played in it belongs to $\underline{G}_j$, is equal to the degree of belonging of the state to $\underline{G}_j$. This admits also the somewhat misleading wording that $\pi_{Xis\underline{G}j}(\omega_i)$ is the *possibility* of the state ω_i, given that the players find themselves involved in the game $\underline{G}_j$. (We say "misleading" because this possibility depends on the fuzzy constraint representing our information that "X is $\underline{G}_j$". Once we are aware that this possibility depends on the condition that ω_i belongs to $\underline{G}_j$ to a certain degree, we can use it without hesitation anymore). If all we want to represent by a possibility function is the fuzzy constraint just mentioned, and no more than that fuzzy constraint, we can also compute the possibility of an event $\underline{G}_j$ by the following property of possibility theory :

$\Pi(\cup_{i\in I} A_i) = \sup \Pi_{i\in I}(A_i)$ for any finite sequence of natural numbers I (where Π stands for possibility)

that is in our case

$$\Pi(\underline{G}_j)=\sup\{\pi_{Xis\underline{G}j}(\omega_i)|\forall\omega_i\in\Omega\},\quad j=1,2$$

and in practice

$$\Pi(\underline{G}_1) = 0.9$$
$$\Pi(\underline{G}_2) = 0.8$$

This means that the overall possibility that event $\underline{G}_j$ occurs is the *supremum* of the set of possibility degrees attached to any state of the world conditioned on the fuzzy information that in the occurring state, whichever it may be, also the event $\underline{G}_j$ does occur.

Let now define the strategic decision problem the players face in this game. First they observe the state of the world selected by Nature. Then they have to decide simultaneously amongst the four strategies depicted in games G1 or G2. Their strategy spaces are $\forall i\ S_i = \{33, 50, 66, D\}$, i=1,2,3,4. They face a decision under vagueness, for the meaning of the strategies D is *vague*. In particular each player considers the case that, if he chooses D, and the game is G2, he may successfully defect and to take advantage of the co-operation of the second players without doing his own part, but alternatively if the game is G1 he may get nothing. Secondly, he considers the case that whenever a proposal of sharing is advanced, the adversary could successfully defect, reaping the entire sum if the game is G2, or alternatively he may fail, leaving both with zero payoffs if the game is G1. These cases are assessed according to the possibility that the state ω_i is selected such that the game G1 or the game G2 is to be played – that is in terms of the respective fuzzy membership functions defined over unforeseen states. Under G1 and G2 it is shared knowledge that solutions theories $\sigma 1$ and $\sigma 2$ are respectively employed. Let us assume that this is a deductive consequence of the theory of games as applied to the two games G1 and G1 that the players take for granted (consider all that as part of the "knowledge base" the players have in mind). As a matter of consequence they attempt to understand whether it is correct to play each of the two solution concepts contingent upon the vague states of the world. Thus their decision set shrinks to {50, D}.

To explain, take the point of view of player P1 and assume that Nature chooses state ω_2. He knows that, if state ω_2 occurred, solution $\sigma1$ is conformed to by the other player at degree 0.5, and solution $\sigma2$ is conformed to by the other player at degree 0.4. In other words his knowledge is representable by possibility 0.5 that the state ω_2 occurs such that the game G1 is going to be played, and possibility 0.4 that state ω_2 occurs such that the game G2 is going to be played. Thus, whenever in the current vague game situation characterised by the state ω_2 he was to play D, his payoff would be 0 under game G1 (with possibility $\pi_{Xis\underline{G1}}(\omega_2) = 0.5$) and 0.21 under game G2(with possibility $\pi_{Xis\underline{G2}}(\omega_2) = 0.4$). On the other hand, whenever when ω_2 occurs he was to play 50 (which is the only other strategy he can choose in order to conform to a solution concept), he would get payoff 0.5 under game G1 (with possibility $\pi_{Xis\underline{G1}}(\omega_2) = 0.5$) and payoff 0 under game G2 (with possibility $\pi_{Xis\underline{G2}}(\omega_2) = 0.4$).

5. Default reasoning

The deliberative procedure used by players is a kind of default reasoning based on fuzzy membership functions mapping states into the fuzzy domain of norms. Reiter (1980) by defaults means rules of inference like syllogisms, which extend the set of statements proved under a given theory and knowledge base by adding to them new statements derived through the application of default rules onto the basic set of statements of the theory and its default consequences. Given a incomplete collection of fact over the world, defaults are a way to "complete" our belief system on the world by inferring what it is allowed by basic beliefs (justified by our knowledge base) plus a set of "reasonable" conditionals that cannot be falsified given our incomplete knowledge. Typical default rules have the following form

$$\frac{A(x)\text{: } MA(x) \rightarrow B(x)}{\therefore\ B(x)}$$

where A(x) is a precondition to the rule belonging to the knowledge base, MA(x) $\rightarrow$ B(x) is the conditional clause that is checked for consistency with the existing base of knowledge (and is assumed by default) and B(x) is a consequence that is added to the base of justified beliefs if the clause is "consistent". According to the coherence interpretation of default logic (Reiter 1980), in fact the modal operator M means "it is consistent to assume that…" and clause MA$\rightarrow$ B can be interpreted as follows: "in the absence of proof to the contrary it is coherent to assume that A$\rightarrow$ B …". (M could also be understood as "normally…" or as "according to our best knowledge of the matter, it is reasonable to think that…").

This illustrates the basic intuition of the logic modelling default reasoning. Default rules of inference permit to add more sentences to our base of knowledge and beliefs by assuming that any conditional is acceptable whenever we have an incomplete collection of positive examples of it and we have not a constructive proof of the contrary to the conditional itself ensuing from the knowledge base. In absence of a refutation, notwithstanding that we do not have a conclusive proof of its truth but only some positive examples, we are permitted to assume the conditional sentence as part of the premises of an inference rule. By using the sentences of a knowledge base - a theory - as major premises of a syllogism, and by adding to them default conditionals, then we derive new conclusions that adds to the theory. In other words, as far as a proof of the contrary doesn't result, the "consistent" clause put together with the base knowledge allows to derive statements which shall constitute extensions of the basic theory. Of course default reasoning is *non-monotonic* and its conclusions are *defeasible*: as more information come out, some conclusion can be retracted in order to account for new information. This is what is shown by the typical case where A(x) stands for *Bird*(x), B(x) for *Fly*(x) and it is known that *penguin*(x) implies *Bird*(x). The consequence of the default inference is "penguin flies", which eventually proves to be false.

To understand how defaults operate to extend a given knowledge base or body of justified beliefs by adding to them new sentences - derived through the application of default rules of inference - some definition are in order. Define a default rules of inference any expression of the following form

$$\frac{\alpha(x):M\beta_1(x),\ldots,M\beta_n(x)}{w(x)} \qquad (\#)$$

where $\alpha(x)$, $M\beta_1(x),\ldots,M\beta_n(x)$, $w(x)$ are first order formulas. $\alpha(x)$ is a premise belonging to the base of knowledge, the $M\beta_i(x)$ are default clauses, $w(x)$ is a consequence, and $x = x_1,\ldots,x_n$ are variables (which are not free under the foregoing expression of a default). Next, following Reiter (1980), define L a first order language, S a set of formulas, w a formula, $\underline{S}$ a set of closed formulas, that is formulas of L without free variables. Finally, define $Th_L(\underline{S})$ the set of closed formulas $w \in L$, which may be proved ("theorems") from the premises $\underline{S}$.

A closed default theory $\Theta = \langle \Delta, \Phi \rangle$ is then a pairs where Δ is a set of defaults of the given form (#), within which any formula is a closed formula of L, and Φ is a set of first order closed formulas of L, belonging to the knowledge base.

For *any* set of closed formulas $\underline{S} \subseteq L$, define an operator Γ such that $\Gamma(\underline{S})$ is the smaller set of formulas satisfying the three following properties

1) $\Phi \subseteq \Gamma(\underline{S})$

2) $Th_L(\Gamma(\underline{S})) = \Gamma(\underline{S})$

3) If $(\alpha:M\beta_1,\ldots,M\beta_n /w) \in \Delta$, $\alpha \in \Gamma(\underline{S})$ and $(\neg\beta_{1,\ldots,}\neg\beta_n) \notin \underline{S}$, then $w \in \Gamma(\underline{S})$

That is, the set $\Gamma(\underline{S})$ includes all the consequences following from premises α included in $\Gamma(\underline{S})$ by defaults, which are consistent with the formulas set $\underline{S}$. Because $\Gamma(\underline{S})$ includes all the basic first order formulas W and the logical consequences of any formulas in $\Gamma(\underline{S})$, this means that $\Gamma(\underline{S})$ contains all the consequences of defaults consistent with $\underline{S}$.

Now take a set of closed formulas $E \subseteq L$ such that Γ (E) = E, that is E is the *minimum fixed point* of the operator Γ. This is what Reiter calls an *extension* for the theory $\Theta = \langle \Delta, \Phi \rangle$. Of course, E satisfy the conditions

1') $\Phi \subseteq$ E

2') $Th_L(E) = E$

3') If $(\alpha{:}M\beta_1,\ldots,M\beta_n /w) \in \Delta$, $\alpha \in E$ and $(\neg\beta_{1,\ldots,}\neg\beta_n) \notin \underline{E}$, then $w \in E$

Thus an extension E contains all the closed formulas derivable by defaults from the theory $\Theta = \langle \Delta, \Phi \rangle$, in such a way that any set of formulas derivable by applying defaults rules to closed formulas belonging to the theory Δ or to closed formulas derived from them, are already included in E.

Let see what amounts to characterising extension E as fixed point (Reiter 1980, Ginsberg 1986). Consider the operator Γ_Δ that carries a set of closed formulas (a formal theory) E to a new set of closed formulas (an extended formal theory) in which the set of default rules Δ have been applied in a way consistent with E. Through Γ_Δ we generate all the default consequences of the set of closed formulas E. Then, because E is already an *extension* of the basic theory $\Theta = \langle \Delta, \Phi \rangle$, it must be such that under the application of the entire set of consistent default rules Δ *it remains unchanged* - that is E contains already every changes that can be effectuated by Γ_Δ so that it is a fixed point of the operator Γ_Δ

$$E = \Gamma_\Delta(E)$$

One important aspect of this definition is that it captures some sense of stability of the theory. The set of believable sentences of an agent under any iteration of the same set of default rules (the operator just defined) are unchanged if his theory is in fact an extension. An extension is the set of sentences defining a formal theory, enunciated by means of the formulas of the formal language of the theory, that cannot be changed by means of the same default rules of inference which was employed to generate the extension and thus remains stable.

6. A first account of the players' default equilibrium selection process

Our main point is that players' reasoning about how to act in unforeseen states of the worlds will proceed by default reasoning in jumping from what they know to what they really don't know but nonetheless is reasonable to assume about the unforeseen states. In our context a default rule of inference is based on the fuzzy information the player have about the truth of various sentences in the unforeseen states of worlds . Notice that we introduce the following shift of the default clauses' meaning. The condition "Given our best knowledge, until the proof to the contrary is given, it is consistent to assume that…" is substituted by "If vagueness doesn't exceeds a given threshold, it is consistent to assume that …". In fact it has the same meaning in the fuzzy context than the typical coherence condition on a default clause. Some statements being not clearly true, nevertheless have not been proved to be false. At least we are vague about them. Moreover, because evidence in their favour keeps vagueness under a given level, and they are not definitely false, we accept them as basis for deriving some believable consequences. We define a threshold by an α-cut set, including all the states that have membership function in $\underline{G}j$, j=1,2, at least equal to the degree δ. Then we take for granted that if a state is member to the set $\underline{G}j$ at least to degree δ, then the state definitely belongs to the domain of the relevant solution concept.

Assume δ = 0.7, then the threshold of vagueness not to be exceeded is 1-δ = 0.3. The δ-cut set defining the event that we can take for granted to be in the domain of application of the solution concept 1 is

$$G1_\delta = \{(\omega_1, 1), (\omega_2, 0), (\omega_3, 0), (\omega_4, 0)\}$$

while the event that we can take for granted to be in the domain of the solution concept σ2 is the δ-cut set

$$G2_\delta = \{(\omega_1, 0), (\omega_2, 0), (\omega_3, 0), (\omega_4, 1)\}$$

Note that these are not fuzzy set but crisp set: one state belongs to any of them or not in a clear cut manner. This suggest also where the default inference come in. We have only to assume that if the states ω_i is a member of the α-cut set $G_{j\delta}$, j=1,2, that is enough to conclude that the game to be played in state ω_1 is G1 and in ω_2 is G2 and the solution concept to be used is the relevant σj for j =1,2. To see how this can be done, let encode our game's statements as follows:

$\alpha_1(\omega_i)$: = "the state ω_i belongs to the event $G1_\delta$"

$\alpha_2(\omega_i)$: ="the state ω_i belongs to the event $G2_\delta$"

$M\beta_1(\omega_i)$: = "if the state ω_i belongs to $G1_\delta$ then 'normally' the game played under the state ω_i is G1"

$M\beta_2(\omega_i)$: = "if the state ω_i belongs to $G2_\delta$ then 'normally' the game played under the state ω_i is G2"

$M\beta_3(\omega_i)$: = "if the state ω_i belongs to $G_{i\delta}$ then normally the solution of the game is σj"

$w_j(\omega_i)$: = "the solution to be played under the state ω_i is σj"

Then the default rule of inference defined for our situation is the following

$$\frac{\alpha_k(\omega_i)\colon M\beta_1(\omega_i), M\beta_2(\omega_i), M\beta_3(\omega_i)}{\therefore\ w(\omega_i)}$$

where formulas $\alpha_k(\omega_i)$ (k =1,2), belonging to the knowledge base Φ of the game, stands for $\alpha_1(\omega_i)$ or $\alpha_2(\omega_i)$ but not both. The first default clause, according to the meaning of the operator M, can be read as follows: "games G_i played in situations ω_i , satisfying the condition $\mu_{\underline{G1}}(\omega_i) \geq \delta$, are normally understood as exemplars of the game class G1". The operator M means also "coherent", i.e. it is coherent to our knowledge to treat the state ω_i as an exemplar of the game class G1, whenever vagueness about the belonging of state ω_i to the event $\underline{G}1$ is positive, but doesn't exceed the stated threshold. Similarly read the second default assumption as refereed to the case when the condition $\mu_{\underline{G2}}(\omega_i) \geq \delta$ is satisfied. Finally the third default assumption is that "games G_j played in situations ω_i satisfying one of the two conditions $\mu_{\underline{Gi}}(\omega_i) \geq \delta$ stated above, are normally understood to be solved according the relevant solution concept σj". According to the foregoing section we may write (α_k: $M\beta_1$, $M\beta_2$, $M\beta_3$ /w_j) ∈ Δ to denote that the three default clauses generate the default knowledge base of our theory. Assume Nature selects the state ω_1, then the default inference rule implies w_1, which says that the solution of the game played in ω_1 is σ1. On the contrary if Nature selects ω_4 we infer by default w_2, which says that the solution σ2 is played.

Application of this first default rule to the belief set of the players simplify considerably their task, because now at least in two states of the world, ω_1 and ω_4 , they jump to the conclusion that there is a definite solution concept. Note that we are assuming that fuzzy sets are the same amongst the two players because vagueness is not a kind of subjective uncertainty but an inter-subjective state of incomplete knowledge (this make far less questionable this assumption with respect to the probabilistic counterpart – the "Harsanyi's doctrine" of a unique common prior). Thus both the players jump to the same conclusion in the same two states and they come identically to treat these two states as they were exactly cases of the typical G1 and G2 games. In these game there is shared knowledge of the appropriate solution theory, and as a matter of consequence the player will employ 50 or D in the two states respectively. We have a simple extensions of those solution theories to cover these two "new" states of the world. If no more information enters the scene (remember that default inferences generate beliefs which are *defeasible* and non monotonic) the two players will both come to the conclusion of having get to the typical equilibrium solutions of the two standard situations.

This is quite a direct way of generating equilibrium solution for games under unforeseen contingencies. New states of the world are simply subsumed under the domain of given general norms (and solution theories) by means of a default inference based on fuzzy pattern recognition of "new" states in terms of the domains of application of "old" general abstract norms. This is only a bit deeper explanation than assuming that in any situation there is a norm which is salient and which is shared knowledge to be complied with by players. In fact we model the cognitive process by which norm can subsume unforeseen states under their domain so that *salience* of a given equilibrium is not an assumption or a brute data of psychology anymore, but begins to be endogenously explained. In fact it is produced by a default rule, applied onto the belief set of the two players, based on fuzzy information about the domain of application of general norms.

However this approach doesn't account properly for the default reasoning process leading to equilibrium selection in the ambiguous game situation under consideration. First of all, it fails to investigate how players manage to get to a unique equilibrium point under the remaining two states of world ω_2 and ω_3, where the conditions for an δ-cut are not satisfied and players cannot jump by this default rule to a definite conclusion about the solution theory to be adopted. This suggests that default rules should not be attached only to high degrees of membership (thresholds), but to any measure of fuzzy membership of the states to the possible game G1 and G2 (or any possibility value of the states as constrained by fuzzy information about their being cases of the game G1 or G2).

Secondly, there is no reason, even if we have considered an example in which this is not the case, for the threshold condition $\mu_{\underline{Gj}}(\omega_i)\geq \delta$, i=1,2,3,4 and j=1,2, must not be satisfied by the same state ω_i for both the crisp events $G1_\delta$ and $G2_\delta$. Hence, two simultaneous default inferences would be admitted together leading to the contradictory conclusions that in the same state ω_i the solution to be played is $\sigma 1$ and the solution to be played is $\sigma 2$. Of course, far from solving the equilibrium selection problem, it would create one more. This also suggests to attach default rules not to high degrees of membership but to the calculations of a player's possibilistic utility, which he can perform in any state, based on the composition of fuzzy utility of actions in a state, under the hypothesis that a given game is to be played in that state, and possibility of the state (derived by the fuzzy information about that state being a case of a given game). Once a player has performed this calculation under the current unforeseen state of the world, it would establish a model of reasoning useful in order to derive the solution strategy that any "normal" player would play in the current unforeseen state of the world. Assigning by a default rule this model of reasoning to any player in the game, under the hypothesis that he is a "normal" player, would permit to derive the possibility that any "normal" player must assign to any other "normal" players' choices and would enable to restart his calculation of the possibilistic utility attached to any choice, given the new possibilistic information. This looks like the typical outguessing regress modelled by some equilibrium selection procedures where each player simulates any layer n of the other players' reasoning through introspection on his own reasoning at layer $n-1$ - see for example Harsanyi and Selten's *tracing percedure* (Harsanyi-Selten 1988, Sacconi 1995).

Third, until now we have not been able to provide a unified language within which to treat both statements concerning the fuzzy knowledge that the players have on states of the world, and statements concerning what the players conclude by default rules of inference about the strategy to be played in each state. This is unfortunate as the premises of the default inference rule are given by a condition expressed in the language of fuzzy membership functions but clauses and consequents of the default inference apparently belong to a quite different language. A remedy to this drawback is provided by the unified language of possibility theory (see sec.7) as it may be employed to account for default reasoning. On the one hand, possibility measures over states, seen as exemplars of certain games, are derived from statements about the players knowledge, i.e. membership functions associating the same states to games meant as fuzzy sets. On the other hand default rules, connecting statements on players' (fuzzy) knowledge of states to statements on players' actions, induce possibility measures over players' choices and over their use of solution concepts associated to types of games, given their knowledge of the current states. Both possibility measures are involved in the same fuzzy utility calculations, which leads any player to select his best response at any step of the equilibrium selection reasoning process.

7. A possibilistic account of the equilibrium selection default reasoning

7.1. Default reasoning and possibility theory

Pieces of incomplete knowledge are described by possibilistic logic formulas, i.e. first order logic formulas with a numerical weight between 0 and 1, which is a lower bound on a possibility measure Π (Dubois&Prade 1996). This is a basic point of our model. For this reason, before going through the description of the players' step in the reasoning process, we give a brief introduction to possibility theory in addition to the fuzzy definition of possibility given in sec.4. Here we are focused on the meaning of some axioms and some propositions fitting our purposes (for more detail see [Benferat et al. 1992; Benfherat et al. 1997; Dubois&Prade 1995; Dubois&Prade 1996]).

As we already know, possibility theory is based on the notion of a *possibility distribution* π which is a mapping from the set U, set of the interpretations of our propositional language, to the interval [0,1] and thus provides a complete ordering of interpretations where the most plausible ones get the highest value 1. Roughly speaking a possibility degree $\Pi(\phi)$ of a formula ϕ, given the set of interpretation which verify this formula, evaluates to what extent ϕ is consistent with the available knowledge expressed by π. Π satisfies the characteristic property:

$$\forall\phi, \forall\psi, \Pi(\phi \vee \psi) = \max(\Pi(\phi), \Pi(\psi)) \; ; \; \Pi(\perp) = 0;$$

where $\perp$ is any inconsistent formula.

Moreover, from the axioms governing possibility measures we have directly that

$$\Pi(\phi\wedge\psi) \leq \min(\Pi(\phi), \Pi(\psi))$$

An ordinal conditioning notion can be defined by means of the Bayesian-like equation

$$\Pi(\phi \wedge \psi) = \min(\Pi(\psi|\phi), \Pi(\phi)) \text{ when } \Pi(\phi) > 0.$$

$\Pi(\psi|\phi)$ is then defined as the greatest solution to the previous equation in accordance with the minimum specificity principle. It leads to

$$\Pi(\psi|\phi) = 1 \text{ if } \Pi(\phi \wedge \psi) = \Pi(\psi) \quad (i.e., \text{ if } \Pi(\phi \wedge \psi) \geq \Pi(\phi \wedge \neg\psi)\});$$

$$\Pi(\psi|\phi) = \Pi(\phi \wedge \psi) \text{ otherwise } (i.e., \text{ if } \Pi(\phi \wedge \psi) < \Pi(\phi \wedge \neg\psi)\}) \quad \text{when } \Pi(\phi) > 0.$$

If $\Pi(\phi) = 0$, then by convention $\Pi(\psi|\phi) = 1, \; \forall \psi \neq \perp$.

Since $\Pi(\phi)>0$, (Benfherat et al. 1997) we have also that

$$N(\neg\psi \mid \phi)>0 \Leftrightarrow \Pi(\phi \wedge\neg\psi)>\Pi(\phi \wedge\psi)$$

where the dual necessity (or certainty) degree $N(\phi) = 1 - \Pi(\neg\phi)$ evaluates to what extent ϕ is entailed by the available knowledge. Therefore we have:

$$\forall\phi, \forall\psi, N(\phi \wedge \psi) = \min(N(\phi), N(\psi)).$$

Finally, regarding the deductive processes involved in possibilistic logic, we want to recall just the following deduction rules, which can be been proved sound and complete for refutation.

If p, q and r are interpretation of our language, and α,β are weight inducing an ordering among the interpretations, the following *resolution rule* yields:

$$\frac{(\neg p \vee q, \; \alpha) \quad (p \vee r, \; \beta)}{(q \vee r, \; \min(\alpha,\beta))}$$

This rule will be employed in order to find a possibilistic interpretation of the a *conditional knowledge base* (i.e., as we will explain widely in the following, a set of *defaults*). In particular it will be employed to find an ordering on the interpretation of the possible world given by the defaults of the players.

7.2. First step in the equilibrium selection process: calculating the first hypothetical best response of a player

We now undertake the possibilistic approach to the study of the default equilibrium selection reasoning in a game situation where nature selects unforeseen states of the world, so that it is vague which of two games G1 and G2 have to be played. We confine our analysis to unforeseen states ω_2 and ω_3 (see sec.4), as they have not been accounted for in the foregoing section. Thus the relevant events, meant as domains of the two solution concepts σ1 and σ2, are the two fuzzy sets $\underline{G}_1=\{(\omega_2, 0.5), (\omega_3, 0.2)\}$ and $\underline{G}_2=\{(\omega_2, 0.4), (\omega_3, 0.4)\}$. We start by considering the possibilistic expected utility calculation that each player performs given his information on the two prototypal game G1 and G2 and the fuzzy knowledge on states ω_2 and ω_3, in order to derive his first conjecture on the solution strategy to be used.

To begin with, a comment is in order about the form of the players' utility functions. These have the form of fuzzy utility functions intended as measures of the degree to which a decision satisfy a flexible constraint in term of intermediary levels of capability. Let the constraint be expressed by the following requirement "take the largest part of 100 dollars under the feasibility condition that shares gained by the two players must sum up to no more than 100". Utility functions express the rank of capability associated to any decision, conditionally on the other player decision, to satisfy the given constraint. This is expressed by means of a membership function, which maps the decision set on the real line [0,1] representing levels of satisfaction. Hence, utility values coincide to membership degrees of a fuzzy set defined over the strategy set of each player, intended as the typical reference set. These utility functions have the appropriate form in order to enable performing the following exercise of possibilistic decision making (they have essentially the same form as the possibility distribution, but defined on the strategy set).

Possibilistic expected utility takes here the form of the max-min operation, given that the operation min is the one appropriate for representing the intersection or conjunction (logical multiplication) of fuzzy sets, and the max operation is the one appropriate for representing the set theoretic union or disjunction (the logical sum) of fuzzy sets. In fact possibility and utility are here coincident to fuzzy sets (or fuzzy distributions) defined the first over the set of unforeseen states of the world and the second over the strategy set of each player. Multiplying the possibility of any event time the utility of a decision when that event occurs, means in this context operating the set-intersection between two fuzzy sets. Summing the expected utility of a decision over different possible events means calculating the set union operation amongst fuzzy sets. Consider one state of the world at time, starting from ω_3 according to the point of view of player P1. Thus player P1's expected utility of strategy D, given the possibility that in state ω_3 game G1 or game G2 are played, is as follows

$$U_1(D \mid \omega_3) = \max\{\min[\pi_{X is \underline{G}1}(\omega_3), u_1(D|G1)], \min[\pi_{X is \underline{G}2}(\omega_3), u_1(D|G2)]\}=$$
$$= \max\{\min(0.2, 0), \min(0.5, 0.21)\} = 0.21$$

where $u_1(D|G1)$ is player P1's utility to play the solution strategy D under the hypothesis that the game played is G1, $\pi_{X is \underline{G}1}(\omega_3)$ is the possibility of state ω_3 under the fuzzy information that the game G_j (i.e. the variable X ranging over the set Ω of possible descriptions of interaction situations) played in ω_3 is G1 and $U_1(D \mid \omega_3)$ is player P1's possibilistic expected fuzzy utility to play D given that ω_3 occurred.

In analogous way we obtain player P1's expected utility of strategy 50

$U_1(50|\omega_3)$=max{ min[$\pi_{X is \underline{G}1}(\omega_3)$, $u_1(50|G1)$], min[$\pi_{X is \underline{G}2}(\omega_3)$, $u_1(50|G2)$] }=
= max{min(0.2, 0.5), min(0.5, 0.21)}=0.2

No other strategies are allowed by the solution theories. In order to maximise his expected utility function contingent on ω_3, player P1 has to select the pure strategy D.

Now consider the case of state ω_2. Player P1's expected utilities for the two strategies prescribed by the two solution theories now are respectively

$U_1(D \mid \omega_2)$ = max {min[$\pi_{X is \underline{G}1}(\omega_2)$, $u_1(D|G1)$], min[$\pi_{X is \underline{G}2}(\omega_2)$, $u_1(D|G2)$] } =
= max {min (0.5, 0), min (0.4, 0.21) }= 0.21.

$U_1(50 \mid \omega_2)$ = max{min[$\pi_{X is \underline{G}1}(\omega_2)$, $u_1(50|G1)$], min[$\pi_{X is \underline{G}2}(\omega_2)$, $u_1(50|G2)$]} =
= max {min (0.5, 0.5), min (0.4, 0) } = 0.5.

In order to maximize his expected utility function contingent on ω_2, player P1 has to select the pure strategy 50.

Thus, we know that if Nature selects state ω_2 or ω_3, then player P1 at the first step of his reasoning process - having at disposal only the information concerning the possibility that states display certain games - must select 50 or D respectively. This is only his "first glance reasoning" however. It doesn't account for the fact that if a solution strategy has to be rationally chosen, it must be a best response against the simultaneous choice made by the other party, who face a similar problem.

7.3. Second step in the equilibrium selection process: inferring by default the second player's reasoning and generating an overall conjecture of his choice

The second step of player P1 reasoning process is his attempt to guess the first step in the reasoning process just executed by the counterpart. Being reflective and rational, even if limitedly so, player P1 undertakes the endeavour of simulating by introspection player P2's reasoning process. Here default reasoning enters again the scene. Player P1 has at disposal his own model of reasoning at the first step. There is no evidence that this is also the first step in the reasoning process of player P2. However he only knows that in two states of the world ω_2 and ω_3 a typical player (himself) reasoned according to the maximisation rule just mentioned and derived the solution theories to be rationally used in these two cases. No proof of the contrary does exist that any other similarly limited rational players would not employ similar calculations. Of course such information could come later on, but as long as the information the player has is the one just described within his model of reasoning, player P1 (at moment) doesn't obtain such contrary evidence. Thus, player P1 introduces a default rule of inference, by assuming that it is coherent with his base of knowledge that player P2 in states like ω_2 and ω_3 will perform exactly the same calculations he performed at his first step in the reasoning process.

Let us define a propositional language by encoding the statements of our theory in the following formulas:

- Φ := the first order knowledge base, which encodes statements concerning

a) the objective description of the states ω_2 and ω_3, including their membership functions to the two fuzzy events $\underline{G}_1$ and $\underline{G}_2$;

b) the algebraic method to calculate fuzzy expected utility;

c) the description of the two solution theory σ_1 and σ_2

- s := "the strategy D being chosen when the state ω_3 takes place *and* the strategy 50 being chosen when the state ω_2 takes place". It is player P1's first step scheme of behaviour, and it may be formally written as $[(\omega_2 \wedge 50) \wedge (\omega_3 \wedge D)]$.

Moreover we encode in our prepositional language the characteristic description of the player i, for $i \in \{1,2\}$, as follows:

- $\Psi(i)$, whose interpretation is "player i owns Φ";
- $\Sigma(i)$, whose interpretation is "player i acts according to s";
- $\Gamma(i)$, whose interpretation is "player i is rational".

Let us consider the following set of default clauses: "if player i owns Φ then normally player i acts according to s", "if player i is rational then normally he owns Φ", symbolically written as the set of default $\Delta = \{\Psi(i) \rightarrow \Sigma(i), \Gamma(i) \rightarrow \Psi(i)\}$, where Δ is the default knowledge base of player P1. Together with the set of formulas constituting the first order knowledge base of our theory Φ, this allows us to define a default theory of the game $\Theta = \langle \Delta, \Phi \rangle$

In order to infer, as a logical consequence of his default theory $\Theta = \langle \Delta, \Phi \rangle$, the decision that will be made by player P2 in each state and his own consequent best response, player P1 needs to combine the calculation of the expected fuzzy utility with his default knowledge base or conditional knowledge. In particular, player P1 needs to deduce the measure of possibility $\Pi_2(50|\omega_2)$ on player P2's choice 50 (i.e. the *plausibility* of the statement "player P2 acts according to s" when the state is ω_2) and, similarly, the measure of possibility $\Pi_2(D|\omega_3)$ on player P2's choice is D. It is in accomplishing this task that the possibilistic logic machinery and his connections to default logic comes in.

The basic idea is to encode each default like "if A then normally B", denoted by $(A \rightarrow B)$, into a constraint expressing that the situation where $(A \wedge B)$ is true has greater plausibility than the one where $(A \wedge \neg B)$ is true (Benferat et al. 1992, Dubois&Prade 1995, 1996). We follow this approach by encoding the defaults denoted by $(\Psi(i) \rightarrow \Sigma(i))$ and $(\Gamma(i) \rightarrow \Psi(i))$ into constraints expressing that the situation where $(\Psi(x) \wedge \Sigma(x))$ and $(\Gamma(x) \wedge \Psi(x))$ are true has greater plausibility than the one where $(\Psi(x) \wedge \neg \Sigma(x))$ and $(\Gamma(x) \wedge \neg \Psi(x))$ are true. Then, we need a qualitative *plausibility* relation $>_\Pi$ for comparing plausibility levels and generating a possibility ranking of situations (Dubois 1986). In the finite case, the only numerical counterparts to the plausibility relations are possibility measures, such that $\Pi(a \vee b) = \max(\Pi(a), \Pi(b))$ for all a, b in the conditional knowledge base. Thus, a default $(\Psi(i) \rightarrow \Sigma(i))$ may be understood formally as an ordinal constraint $(\Pi\Psi(i) \wedge \Sigma(x)) > (\Pi\Psi(i) \wedge \neg \Sigma(i))$ on the possibility measure Π, introduced *via* his qualitative counterpart based on $>_\Pi$, and describing the semantic of the knowledge available to player i. The ordinal constraint on possibility can be also shown to be equivalent to the measure of necessity $N(\Sigma(i)|\Psi(i))$ which estimates to what extent strategies different from s have a low degree of possibility to be chosen by player x given that player i owns $\Psi(i)$.

Following Dubois&Prade (1995) and considering the two defaults in the player P1's default knowledge base, the set of defaults $\Delta=\{\Psi(x) \rightarrow\Sigma(x), \Gamma(x) \rightarrow\Psi(x)\}$ will be represented by the following set of qualitative constraints C:

c1: $\Psi(x) \wedge \Sigma(x) >_\Pi \Psi(x) \wedge \neg\Sigma(x)$

c2: $\Gamma(x) \wedge\Psi(x) >_\Pi \Gamma(x) \wedge \neg\Psi(x)$.

As numerical counterparts to plausibility relations are possibility measure, the qualitative constraints translate into the following set of ordinal constraints on possibility C'

c'1: $\Pi(\Psi(i)\wedge \Sigma(i)) > \Pi(\Psi(i)\wedge \neg\Sigma(i))$

c'2: $\Pi(\Gamma(i)\wedge \Psi(i)) > \Pi(\Gamma(i) \wedge \neg\Psi(i))$.

Let U be the finite set of interpretations of *our propositional language* $\Psi(i)$, $\Gamma(i)$, $\Sigma(i)$, Φ, s, x. These interpretations correspond to artificial possible worlds, worked out by the elements of the artificial propositional language just defined, in which the conjunctions $*\Psi(x)\wedge*\Gamma(x)\wedge*\Sigma(x)$ are true, where $*$ stands for the presence of the negation sign $\neg$ or its absence. Hence the models of our formal language are

U = {u_0: $\neg\Psi(x)\wedge \neg\Sigma(x)\wedge \neg\Gamma(x)$; u_1: $\neg\Psi(x)\wedge \neg\Sigma(x)\wedge\Gamma(x)$; u_2: $\neg\Psi(x)\wedge\Sigma(x)\wedge \neg\Gamma(x)$; u_3: $\neg\Psi(x)\wedge \Sigma(x)\wedge\Gamma(x)$; u_4: $\Psi(x)\wedge\neg\Sigma(x)\wedge \neg\Gamma(x)$; u_5: $\Psi(x)\wedge \neg\Sigma(x)\wedge \Gamma(x)$; u_6: $\Psi(x)\wedge \Sigma(x)\wedge\neg p$; u_7: $\Psi(x)\wedge \Sigma(x)\wedge \Gamma(x)$}.

Then the set of ordinal constraints C' on possibility measures translates into the following set of constraints C" on the possibility order of models:

c"1 : $\max(\pi(u_6), \pi(u_7)) > \max(\pi(u_4), \pi(u_5))$

c"2 : $\max(\pi(u_5), \pi(u_7)) > \max(\pi(u_1), \pi(u_3))$.

Let now $>_\pi$ be a ranking of U, such that $u >_\pi u'$ iff $\pi(u) > \pi(u')$ for each $u, u'\in U$. Any finite consistent set of constraints like $(a \wedge b) >_\Pi (a \wedge\neg b)$ thus induces a partially defined ranking $>_\pi$ on U, that can be completed according to the principle of minimum specificity. The idea is to try to assign to each world $u\in U$ the highest possibility level (in forming a well-ordered partition of U) without violating the constraints. The ordered partition of U associated with $>_\pi$ using the minimum specificity principle can be easily obtained by the following procedure (Dubois&Prade 1995)

a. $i = 0$, $F_i= \{\emptyset\}$

b. As far as U is not empty, repeat b.1.-b.4.:

b.1. $i \leftarrow i + 1$

b.2. Put in F_i every model which does not appear in the right side of any constraints in C",

b.3. Remove the elements of F_i from U

b.4. Remove from C" any constraint containing elements of F_i.

Let us apply now this algorithm. We obtain:

- $i=1$: $F_1=\{u_0, u_2, u_6, u_7\}$, $U=\{u_1, u_3, u_4, u_5\}$, $C"=\{\emptyset\}$

- $i=1$: $F_2=\{u_1, u_3, u_4, u_5\}$, $U=\{\varnothing\}$, $C'=\{\varnothing\}$

Finally, the well ordered partition of U is: $\{u_0, u_2, u_6, u_7\} >_\pi \{u_5, u_4, u_1, u_3\}$.

Notice that $\Sigma(x)\Leftrightarrow u_2 \vee u_7$ and also $\Sigma(x)\Leftrightarrow u_3 \vee u_6$. Since $\Pi(a \vee b)=\max(\Pi(a), \Pi(b))$ and for each $u\in F_2$ it is true that $u_2>_\pi u$, $u_6>_\pi u$, $u_7>_\pi u$, then

$$\Pi(\Sigma(i))\geq\Pi(u\vee u') \text{ for all } u, u' \in U . \qquad \textbf{(1)}$$

In particular also $\Pi(\Sigma(i))\geq \Pi(\neg\Sigma(i))$ is satisfied.

Without loss of generality, we can write $\Pi(\Sigma(i))=\Pi_i(s)$, with $i=1,2$, where $\Pi_i(s)$ is the measure of possibility that player i acts according to $(\omega_2 \wedge 50)\wedge(\omega_3 \wedge D)$. Moreover, from the axioms governing possibility measures we have directly that

$$\Pi_i(s)= \Pi_i[(\omega_2\wedge 50)\wedge(\omega_3\wedge D)] \leq \min[\Pi_i(\omega_2\wedge 50), \Pi_i(\omega_3\wedge D)]$$

and hence directly from (1)

$$\min[\Pi_i(\omega_2\wedge 50), \Pi_i(\omega_3\wedge D)] \geq \Pi(\neg\Sigma(i))= \quad \Pi_i(\neg s) \qquad \textbf{(2)}$$

Consider $\Pi_i(\omega_2\wedge 50)$. From (2) it follows that

$$\Pi_i(\omega_2\wedge 50)\geq\Pi(\neg\Sigma(i))=\Pi_i(\neg s) \qquad \textbf{(3)}$$

Note that, since ω_2 and ω_3 are always true, we can write the following relation

$$\neg s \Leftrightarrow \neg((\omega_2\wedge 50)\wedge(\omega_3\wedge D)) \Leftrightarrow ((\omega_2\wedge\neg 50)\vee(\omega_2\wedge\neg D)\vee(\omega_3\wedge\neg 50)\vee(\omega_3\wedge\neg D))$$

Therefore

$$\Pi_i(\neg s)=\max(\Pi_i(\omega_2\wedge\neg 50),\Pi_i(\omega_2\wedge\neg D), \Pi_i(\omega_3\wedge\neg 50),\Pi_i(\omega_3\wedge\neg D)) \qquad \textbf{(4)}$$

From (3) and (4) it follows directly that $\Pi_i(\omega_2\wedge 50)\geq \Pi_i(\omega_2\wedge\neg 50)$. Since $\Pi_i(\omega_2)>0$, (Benfherat et al. 1997) we have also that

$$N_i(\neg 50 \mid \omega_2)>0 \Leftrightarrow \Pi_i(\omega_2\wedge\neg 50)>\Pi_i(\omega_2\wedge 50)$$

where $N_i(\neg 50 \mid \omega_2)$ is the *conditional necessity measure* which estimates to what extent strategies different from s have a low degree of possibility to be acted by player i given that the state ω_2 occurred (Benfherat et al. 1997). Since the second member is false, we conclude that $N_i(\neg 50 \mid \omega_2)=0$.

By definition $N_i(\neg 50\mid \omega_2)= 1-\Pi_i(50\mid \omega_2)$, where $\Pi_i(50\mid \omega_2)$ is the *conditional possibility measure* that player i plays 50 given that the state ω_2 occurred; thus we obtain $\Pi_i(50\mid \omega_2)=1$. In a similar way we obtain also that $\Pi_i(D \mid \omega_3)=1$.

At the end it may be concluded that if player P1 uses the default rules represented by the constraints on the possibility ranking of models given at the beginning of this section, then the possibility he assigns to the choices of player P2 reproduces the behaviour that player P1 would have maintained at step 1 of his reasoning. However, in order to produce a reasonable overall guess of players P2's choices, player P1 has to combine two pieces of vague knowledge, i.e. the first step fuzzy knowledge about each state being and exemplar of any given possible game, and the second step default conclusions concerning player P2's first step choice. A reasonable way to do that is first considering the conjunction of any pair of events like "player P2 chooses strategy 50" and "the game played in the current state is Gj," i.e.$(G_j \mid \omega_h)$, for j=1,2 and h=2,3. Second taking the possibility of the resulting conjoint conditioned events, i.e. $(50\wedge(\underline{G_j}\mid\omega_h))$ - where it happens that, when the state ω_h is the case, the game played is $\underline{G}_j$ and player P2 chooses strategy 50. Given sec.7.1 Bayes-like definition of the possibility of conjoint events, we may write

$\Pi_i[50\wedge(\underline{G}_j|\omega_h)]=\min[\Pi_i[50|(\underline{G}_j|\omega_h)], \Pi_i(\underline{G}_j|\omega_h)],(i=1,2; h=2,3; j=1,2)$

The meaning of $(\underline{G}_j|\omega_h)$ is “occurrence of the game $\underline{G}_j$ in the state ω_h” that, given the vague knowledge context, corresponds to the fuzzy membership function saying how it is vague that state ω_h belongs to the event $\underline{G}_j$ - that is how it is true that when state ω_h occurs also the event $\underline{G}_j$ does occurs. The double conditioning of choice 50 is not troubling because the meaning of the conditioned event $50|(\underline{G}_j|\omega_h)$ is the same as the conditioned event $(50|\omega_h)$ In fact the event that player P2’s chooses 50 can be subordinated by player P1 only to the occurrence of a state $\omega_{h,}$, which is the only evidence he may observe, being definitely vague which game is to be played in ω_h. Moreover because the membership function represents an inter-subjective state of vague information, the same for all the players, it follows that we may write

$$\Pi_i(\underline{G}_j|\omega_h)= \mu_{\underline{G}j}(\omega_h) = \pi_{Xis\underline{G}j}(\omega_h)$$

Thus we obtain

$$\Pi_i[50\wedge(\underline{G}_j|\omega_h)] = \min[\Pi_i(50|\omega_h), \pi_{Xis\underline{G}j}(\omega_h)]$$

Then, we may calculate

$\Pi_i[50 \wedge (G_1|\omega_2)]= \min[\Pi_i(50|\omega_2), \pi_{XisG1}(\omega_2)]= \min(1, 0.5)= 0.5,$

as a possibility measure that player i plays 50 in the game G_1 given the state is ω_2, and for the remaining Joint events

$\Pi_i[(D \wedge(G_1| \omega_3)] = \min[\Pi_i(D| \omega_3), \pi_{Xis\underline{G}1}(\omega_3)] = \min(1, 0.2) = 0.2,$

$\Pi_i[(50 \wedge (G_2| \omega_2)] = \min[\Pi_i(50|\omega_2), \pi_{XisG2}(\omega_2)] = \min(1, 0.4) = 0.4$,

$\Pi_i[(D \wedge(G_2| \omega_3)] = \min[\Pi_i(D| \omega_3), \pi_{Xis\underline{G}2}(\omega_3)] = \min(1, 0.5) = 0.5.$

These possibility measures conclude player P1’s attempt to guess the reasoning the player P2 performs at the first step of his reasoning process.

7.4. Third step in the selection process: calculating a second hypothetical best response

Player P1at third step of his reasoning process may now check whether the conclusions he arrived to in the first step, relative to the strategy to be used in each unforeseen state ω_2 and ω_3, are stable against the new beliefs he has gained by default about player P2’s action in each state.

A reasonable way to calculate expected utility for player P1 is

$U_1(D|\omega_2) =$

$= \max\{\min\{\Pi_2[(50\wedge G_1)|\ \omega_2],u_1[D|(G_1\wedge 50)]\},\min\{\Pi_2[(50\wedge G_2)|\ \omega_2],u_1[D|(G_2\wedge 50)]\}\}=$

$= \max\{\min(0.5 , 0), \min(0.4 , 1)\}= 0.4$

where $u_1[D|(G_1\wedge 50)]$ is the utility value of player P1 when player P2 plays 50 in the game G_1. Similarly

$U_1(50|\omega_2) =$

$=\max\{\min\{\Pi_2[(50\wedge G_1)|\ \omega_2],u_1[50|(G_1\wedge 50)]\}, \min\{\Pi_2[(50\wedge G_2)|\ \omega_2],u_1[50|(G_2\wedge 50)]\}\}=$

$=\max\{\min(0.5, 0.5), \min(0.4 , 0.5)\}= 0.5$

$U_1(D|\omega_3) =$

$=\max\{\min\{\Pi_2[(D\wedge G_1)|\ \omega_3],u_1[D|(G_1\wedge D)]\}, \min\{\Pi_2[(D\wedge G_2)|\ \omega_3],u_1[D|(G_2\wedge D)]\}\}=$

$=\max\{\min(0.2 , 0), \min(0.5 , 0.21)\}= 0.21$

$U_1(50|\omega_3)=$

$\max\{\min\{\Pi_2[(D\wedge G_1)|\ \omega_3],u_1[50|(G_1\wedge D)]\}, \min\{\Pi_2[(D\wedge G_2)|\ \omega_3],u_1[50|(G_2\wedge D)]\}\}=$

$=\max\{\min(0.2 , 0), \min(0.5 , 0)\}= 0$

The expected fuzzy utility to play 50 when the state ω_2 occurs is greater than the expected fuzzy utility to play D when the same state occurs, whereas the fuzzy utility to play D when the state is ω_3 occurs is greater than to play 50 when ω_3 occurs. Remember that at step two player P1 stated a set of default rules (the conditional knowledge base Δ) in order - as far as his understanding of the matter was concerned - to account for the reasoning of a “normal” rational player. Now he has to check whether the conclusions he reached at step three (by calculating his “second guess” about his highest fuzzy expected utility strategy given his prediction of player P2’s choices), are not inconsistent with the default model of “normal” reasoning. Notice that a default theory is not monotonic. The third deductive step in the player P1 reasoning process could produce new conclusions about normal rational reasoning that could be incompatible with the conclusions derived by default at his foregoing steps. This were the case, he would be requested to retract some of conclusions derived from his default base of knowledge Δ. This sums up in checking whether the possibility measures derivable from the last calculation of his best reply are consistent with the constraints laid on possibility ordering by the defaults contained in the default knowledge base Δ.

To begin with, we can argue that the conditional measure of possibility that player P1 plays 50 given that the state ω_2 occurs is higher than the conditional measure of possibility that player P1 plays D given that the same state occurs. This is a natural consequence of the reasoning accomplished by player P1 at his third step. So we can say in formulae

$$\Pi_1(50 \mid \omega_2) > \Pi_1(D \mid \omega_2)$$

Since $\Pi_1(50 \mid \omega_2) = 1\text{-}N_1(\neg 50 \mid \omega_2)$ and $\Pi_1(D \mid \omega_2) = 1\text{-}N_1(\neg D \mid \omega_2)$, then

$$N_1(\neg D \mid \omega_2) > N_1(\neg 50 \mid \omega_2)\geq 0.$$

Moreover, we note that $N_1(\neg D \mid \omega_2)>0 \Rightarrow \Pi_1(\omega_2\wedge\neg D)>\Pi_1(\omega_2\wedge D)$.

Since $\omega_2\Leftrightarrow(\omega_2\wedge\neg D)\vee(\omega_2\wedge D)$, then

$$\Pi_1(\omega_2)=\max(\Pi_1(\omega_2\wedge\neg D),\Pi_1(\omega_2\wedge D))= \Pi_1(\omega_2\wedge\neg D).$$

On the other hand, we had already seen from (3) and (4) that $\Pi_1(\omega_2\wedge 50)\geq\Pi_1(\neg s)$, and that $\Pi_1(\omega_2\wedge\neg D)$ is at most as high as $\Pi_1(\neg s)$. It follows that

$$\Pi_1(\omega_2\wedge 50) \geq \Pi_1(\omega_2\wedge\neg D)= \Pi_1(\omega_2) \qquad \textbf{(5)}$$

and by definition

$$\Pi_1(\omega_2 \wedge 50) \leq \min(\Pi_1(\neg 50), \Pi_1(\omega_2)) \leq \Pi_1(\omega_2) \quad \textbf{(6)}$$

Hence, by (5) and (6) we obtain that

$$\Pi_1(\omega_2 \wedge 50) = \Pi_1(\omega_2) \quad \textbf{(7)}$$

Note that, for $\Pi_1(\omega_2)>0$, the conditional possibility measure $\Pi_1(50|\omega_2)$ is defined (Benfherat et al. 1997) as the greatest solution to the equation $\Pi_1(\omega_2 \wedge 50)=\min(\Pi_1(50|\omega_2),\Pi_1(\omega_2))$ in accordance to the minimum specificity principle. Therefore

$$\Pi_1(50|\omega_2)=1 \text{ if } \Pi_1(\omega_2 \wedge 50)= \Pi_1(\omega_2)$$

and

$$\Pi_1(50|\omega_2)= \Pi_1(\omega_2 \wedge 50) \text{ otherwise.}$$

Since by (7) we have $\Pi_1(\omega_2 \wedge 50) = \Pi_1(\omega_2)$, thus $\Pi_1(50|\omega_2) = 1$ and

$$N_1(\neg 50|\omega_2) = 1-\Pi_1(50|\omega_2)=0.$$

As we saw above, it can be easily checked that for $\Pi_1(\omega_2)>0$

$$N_1(\neg 50|\omega_2)>0 \Leftrightarrow \Pi_1(\omega_2 \wedge \neg 50)> \Pi_1(\omega_2 \wedge 50)$$

Since the first member is false, we conclude that

$$\Pi_1(\omega_2 \wedge 50) \geq \Pi_1(\omega_2 \wedge \neg 50) \quad \textbf{(8)}$$

Note also that from (5) we have

$$\Pi_1(\omega_2 \wedge 50) \geq \Pi_1(\omega_2 \wedge \neg D) \quad \textbf{(9)}$$

By similar arguments we can also deduce that

$$\Pi_1(\omega_3 \wedge D) \geq \Pi_1(\omega_3 \wedge \neg D) \quad \textbf{(10)}$$

and

$$\Pi_1(\omega_3 \wedge D) \geq \Pi_1(\omega_3 \wedge \neg 50). \quad \textbf{(11)}$$

Comparing (**8**),(**9**),(**10**),(**11**) with (**4**), i.e. comparing the possibility of actions encoded by s with the possibility of actions encoded by ¬s, player P1 does not find any inconsistency with the defaults contained in the conditional knowledge base Δ. Therefore player P1 has no reason to change his previous conditional knowledge base. Moreover also in this case player P1 must keep playing the strategy s that he calculated as his best choice at the first step in his reasoning process. This means that, for example, as far as player P2 is predicted by default reasoning to play his strategy D in the state ω_3, player P1's possibilistic best response is strategy D, as this is player P1's best response calculated given the first step fuzzy knowledge on states and the second step default knowledge on player P2 choices. Of course (D,D) is the equilibrium pair dictated by the solution theory σ2 of game G_2. However players at this stage do not have definite knowledge that the game to be played in the current state is G_2. Nevertheless this the equilibrium pair which tends to emerge form the reasoning process the players perform in the ongoing game under unforeseen contingencies, when they only know that ω_3 is the case in point. Conversely the strategy pair (50, 50) is the emerging solution when they only know that the state ω_2 is the case in point.

7. 5. Further steps: equilibrium selection

Little work is needed to verify that if by analogous default rules player P1's reasoning at step 3 is assigned to player P2, in order to recalculate his strategy choice at the third step of his reasoning process, player P2 can only be predicted to reach the same conclusions as player P1. In other words player P2 can be predicted by default to be recalculating his fuzzy expected utility given similar defaults over player P1 reasoning and reaching the symmetrical conclusion that strategy 50 is to be played when ω_2 is the case and strategy D is to be played when ω_3 is the case. This would be taken as the fourth step in player P1 default reasoning process. Moreover if player P1 uses this information in order to recalculate his further best response, it is fairly clear that, given data are unchanged, results cannot change with respect to any states of the world ω_2 and ω_3. This is to say that applying iterately the same default rules in order to deducing new extensions of the players' theory of the game and new reasonable beliefs based on default inferences, the set of statements of the theory that the players believe does not change anymore. Let us state this result as the following

PROPOSITION I: *Each player, by replicating the other player's reasoning process with iterated applications of the same set of defaults belonging to Δ, is driven to believe that each of them is going to play the solution strategy-pair $\sigma 1$ (respectively $\sigma 2$) if the current unforeseen state is ω_2 (respectively ω_3) even if he does not have an uncontroversial proof of the truth of the statement that in the state ω_2 (respectively ω_3) the game G_1 (respectively G_2) is to be played.*

This traces us back to an important feature of an *extension* E meant as the consistent set of all the formulas derivable from a given first order knowledge base conjoint to a set of defaults (a conditional knowledge base) - i.e. a default theory $\Theta = \langle \Delta, \Phi \rangle$ - in such a way that any set of formulas derivated by applying default rules to closed formulas belonging to Θ or derivated from them are already included in E. Let consider again the definition of an extension E as (the minimal) fixed point of the default inference operator Γ_Δ

$$E = \Gamma_\Delta(E)$$

where Γ_Δ carries sets of closed formulas to new sets of closed formulas by applying the set of defaults Δ without introducing any inconsistency into them. It applies to our case in the sense that by recursive application of the same defaults to the reasoning of the adversary, in order to generate simulations of the other player's steps in his reasoning process, and hence by using that prediction as a basis for recalculating the first player's own best response, no new sentences of the theory contradicting the old ones are produced, nor changes occur in the calculated best responses, nor them intervene in the set of beliefs each player has about the other player's behaviour.

There is a natural game theoretical interpretation of an extension seen as a fixed point of the default operator applied in order to deduce any internally consistent set of conclusions from the base theory of an agent. Consider that defaults are employed to simulate repeatedly the reasoning of each player by means of the models of reasoning of the other, each being concerned with a prediction of the other's behaviour.

When the players reach an extension of the knowledge base of the game, which is enlarged by the applications of a basic set of defaults, then their beliefs are at a fixed point. Further iterations of the reasoning process, as it is driven by these defaults, will reproduce the same set of beliefs. What emerges is a system of internally *consistent* expectations for each player, where any order layer of a player's belief confirms the lower one. A system of reciprocal expectations that are stable and confirm each other at any level, on the other hand, is an appropriate state of players' beliefs in order to say that, contingent on any given state of the world ω_i, they reasonably believe that one equilibrium point will constitute the unique solution of the game. As a consequence, if it is accepted the hypothesis that both the players reason identically, by resorting to the same set of defaults, they will select the same solution concept relative to the given unforeseen state of the word. Let state this result by the following

PROPOSITION II*: Assuming that any player P_i will use the same default theory of the game $\Theta = \langle\Delta,\Phi\rangle$ and they, in order to guess each other's choice, will employ the default reasoning operator Γ_Δ naturally defined by recursively applying defaults belonging to Δ to sentences in Θ or derivated from Θ,* then *they will eventually converge to play the same solution concept strategies, which is an "equilibrium" (i.e. a state from which they do not have any incentive to move in terms of fuzzy utility) relative to the state of their knowledge represented by the* fixed point *of their default reasoning operator*.

Some remarks are in order to point out how stable this result is. *First*, we are assuming inter-subjective vagueness, so that the players P_i have the same fuzzy sets representations of the relationships between unforeseen states and the *ex ante* defined classes of games G_j. *Second*, being quite obvious that the first order knowledge base is identical among them, we also assume that they share the same default rules. That is the model they employ for guessing the other player reasoning is not only their own model of reasoning attributed by default to the other players, but it is also the same model among them. While the first assumption seems to have some foundation in *fuzzy sets thinking* (Kosko 1993), the second is a simplifying one, which should be relaxed in a more general context.

However this need not to be the more serious fragility of the "equilibrium" to which the players converge. The fact that employing the same default reasoning operator the players eventually reach to an *extension*, which implies stability of the reciprocal guesses and induces the players to consistently adopt the same solution σj, does not imply however that they are *in fact* playing the proper equilibrium-solution strategy (in the Nash equilibrium sense) of the *effective* game situation they are confronted in the current unforeseen state. By recollecting more information they could *ex post* discover that they were mistaken in thinking that for example in state ω_3 strategy D was to be used. Thus the solution concept the players may select as equilibrium of the game under unforeseen contingencies is only a *conjectural* solution concept, in the sense that it is conditional on a given default theory $\Theta = \langle\Delta,\Phi\rangle$ and it is non- monotonic with respect to the flux of facts they may learn about the unforeseen state of the worlds. The possibility to discover that previous beliefs, apparently justified by the "normal" course of facts, are mistaken, is inherent to the non-monotonic nature of default reasoning. Given the extension of their theory derived by default rules, they see a solution concept as the rational way of playing the game under unforeseen contingencies. They do that however without any assurance that *ex post* - when they will have accumulated enough experience of the game played under the current state - they will not learn facts showing that in the current state the true game to be played was quite different.

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Notes

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[2] Second Draft, January 2002
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[5] This result can be put in the format of a paradox. Assume that the starting point of any equilibrium selection process cannot be an equilibrium point itself. Hypothesise that the players have common knowledge that (i.e. it is true that) there is a given common prior which synthesises the uncertainty the players have on the possible equilibrium choices one of the others. Then, each player, knowing the common prior, will adopt a best response strategy, which reasonably will not be described with probability one by the common prior (otherwise we would have been in equilibrium since the beginning of the selection process). It follows that, if there is common knowledge of the common prior, then each player must know that both himself and any other player will adopt a best response strategy to the *common prior-behaviour* of the others, which is not exactly correspondent to the common prior itself – i.e. he knows (and expects) that any other's behaviour deviates from the common prior. But, given the definition of knowledge used in the epistemic logic used for formalizing common knowledge - Aumann (1976), Binmore and Brandenburger (1990), Fagin, Halpern, Moses and Vardi (1995) - this means that for each player the state of knowledge wherein he finds himself *contradicts* that he knows the common prior - i.e. contradicts that the common prior is the *true state of knowledge of any player.* Thus if at the starting point of an equilibrium selection process there is common knowledge of the common prior*, then it cannot be true that the common prior reflects what the players believe about the other players' equilibrium choice in the game.* In other words any "eductive" equilibrium selection process endowed with a common prior must assume limited reasoning based on building-block such as deductive steps based on fallible and retractable conjectures concerning other players expectations and continuous revision of such fallible deductions, until they enter a state of "equilibrium expectations" (Sacconi 1995b).
[6] In fact we need only to account for a less cumbersome state of knowledge, which we call "shared knowledge", such that "everybody knows that in game G the solution is σ, and everybody knows that everybody knows that in game G the solution is σ". Infinite iterations of this basic piece of shared knowledge are not required in order to justify that an equilibrium points is selected in a context of limitedly rational players. Requiring common knowledge in the strict sense would ask for an infinite hierarchy of mutually consistent layers of beliefs or knowledge, and this in turn would imply logical omniscience of the players, which seems an inappropriate assumption to be made in our context.
[7] Notice that in game G2 there are not multiple equilibriums so that in this game *per se* there is not any equilibrium selection problem. However in this paper we generate the typical multiplicity problem in a way quite different from the standard one. We shall assume that there are different possible games, each endowed with a unique equilibrium solution (which in G2 is also in dominant strategies), and the players have ambiguous information concerning the game they are going to play. Hence, they must decide which of the two equilibrium strategies to play.
[8] We may say that vagueness ranges over the questions whether the rules of game necessary to use the Nash bargaining solution do apply or the rules of game necessary to use the typical non-cooperative DP solution do apply.
[9] You may think "as if" the language employed in describing concrete but unexpected properties of the unforeseen states belonging to Ω had undergone to a sort of evolutionary pressure for practical descriptive reasons, so that it is in some extent changed with respect to the old one of general and abstract terms employed to describe the general game forms G1 and G2. Thus there is a sort of ambiguity in classifying new objects expressed by the new evoluted, concrete predicates in terms of the game concepts used to define G1 and G2.