BASIC INSIGHTS IN PRICING BASKET CREDIT DERIVATIVES

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Introduction[1]

Basket credit derivatives are those financial contracts whose payout depends on the credit events ("failure to pay", "default", et cetera) characterising a portfolio of bonds or loans over a determined time horizon.

We have two main categories of basket credit derivatives. The first is characterised by a payout depending on the temporal ranking of the credit events: first-to-default, second-to-default, et cetera. The second is characterised by a payout depending on the percentiles of the portfolio's loss distribution induced by the credit events. The latter is often embedded in securitisations of portfolios of bonds or loans, i.e. CDO.

This paper proposes some basic insights in the pricing of these particularly complex credit derivatives. Whenever possible, we will try to find an analytical approximation to the exact pricing formula, if a closed form solution is not available.

The complexity of modelling the loss distribution of a portfolio of assets has been simplified. Duffie and Singleton, among the others, have showed how to build multi-factor affine models for a credit curve. However, we will remain a step behind the state-of-the-art in analytical modelling. In fact, we haven't addressed the crucial problems of parameters' estimation and we are not sure of the way in which one should treat in practice the uncertainty surrounding those estimates. The reader should then be aware that the proposed solution, apart from being in many cases an approximate one, is just a starting point in the difficult task of assessing the fair value of a basket credit derivative.

1. Definitions

In the following, for the sake of simplicity and without any loss of generality (with respect to the case of a deterministic recovery value), we will assume a zero recovery rate in case of default. This implies that the loss value will be equal to the face value of the asset.

1.1 Rank basket derivative

In this category of credit derivatives we find those contracts generally known as "first-to-default". In its simplest form, a rank basket derivatives states that party A ("protection seller") will pay to party B ("protection buyer") a certain amount of money if and when one of the companies ("names") indicated in the contract is hit by a credit event ("default") before a specified maturity date. The amount of money paid in case of default is in general equal to the principal of the loans or the bonds issued by the defaulted names, as indicated in the contract. In general, the principal is evenly distributed among the names of the basket. The list of admissible credit event/default is produced by ISDA, which is trying to standardise the contracts (but there are still some uncertainties, as the "debt restructuring" case witnesses).

The price of a rank basket derivative covering the first x credit events will be equal to:

$$P(1,2,...,x) = \sum_{i=1}^{x} P(i) = \sum_{j=1}^{x} L_x \int_0^T e^{-rt_x} f(t_x) dt_x \quad (1)$$

where we denote by T the credit derivative maturity, t_x the random variable measuring the time we have to wait until the *x*-th credit event happens, *f(.)* the density function of the waiting time, and L_x the associated loss.

1.2 Percentile basket derivative

In its simplest form, a percentile basket derivative might state that the protection seller will compensate the protection buyer for the losses registered on certain portfolio of bonds, up to the first α%. The losses are obviously determined by a specific set of credit events occurring before the contract maturity date. If α% is 100%, this is a traditional form of guarantee, which covers the protection buyer from all the losses. If α% is 20% and the losses eventually amount to 30%, the protection seller will reimburse only the 20% of the losses, whereas the protection buyer will bear the remaining 10%. In this category of credit derivatives we find the CDO, as we will see in a moment.

The pay-out for the protection buyer will be equal to:

$$\min\{L_a, L\} \tag{2}$$

where L denotes the total losses actually occurred and L_a denotes the α-percentile of the portfolio's loss distribution (recall that with zero recovery rate the loss, in case of default, is equal to the face value of the asset).

Let's then imagine a financial contract which divides in two tranches the final pay-out of a portfolio of bonds or loans. According to this contract, the buyer of the first tranche (the "junior" tranche) will receive at maturity the α% of the total nominal value of the basket less the losses occurred. If the losses exceed the α% of the total nominal value of the basket, the junior tranche is exhausted and the difference between L and L_a is paid by the holder of the second tranche (the "senior" tranche). The holder of the senior tranche will receive (1- α)% of the total nominal value of the basket less the losses in excess of L_a .

In this respect, the holder of the junior tranche can be seen as a protection seller with respect to the holder of the senior tranche.

The pay-out at maturity for the holder of the junior tranche will then be equal to L_a less the (2) pay-out:

$$\begin{aligned} \Pi(0,a) &= L_a - \min\{L_a, L\} \\ &= \max\{L_a - L, 0\} \\ &= Put(L, L_a) \end{aligned} \tag{3}$$

As one can see, this pay-off is similar to the one of a put option on the basket's losses, with strike equal to L_a . The protection seller is, then, buying a put option on the portfolio losses, which can be in or out-of-the-money depending on the expected loss rate.

Using the analogy with the option pricing theory, the lower the expected loss rate, the more the put option goes in-the-money and the higher becomes the price of the first-loss derivatives. In terms of spread over Libor, this implies a lower spread. We can also see that the higher the dispersion of the loss distribution the higher will be the value of the put option, and then the lower the spread required.

The pay-out at maturity for the holder of the senior tranche will be equal to (100- L_a) less the excess losses:

$$\begin{aligned}\Pi(a,1) &= (100 - L_a) - \max\{L - L_a, 0\} \\ &= (100 - L_a) - Call(L, L_a)\end{aligned} \quad (4)$$

As one can see, the second term on the r.h.s. of eq. (4) is similar to the pay-off of a call option on the basket's losses, with strike equal to L_a . Eq. (4) suggests that the senior tranche is equal to being long a zero coupon bond and short a call option on the losses. Clearly, the lower is the loss rate the more out-of-the-money will be the call option (and the better-off the buyer of the senior tranche). At the same time, the higher is the volatility of the loss rate, the higher will be the value of the call option and the lower the value of the senior tranche. The latter implication translates in a higher spread over Libor for the senior tranche.

Since in equilibrium, the sum of the two tranches has to be equal to 100 (we consider the CDO as a zero sum game), the following accounting identity has to hold:

$$\begin{aligned}100 &= \Pi(a,1) + \Pi(0,a) \\ &= (100 - L_a) - Call(L, L_a) + Put(L, L_a)\end{aligned} \quad (5)$$

Eq. (5) can be seen as stating a sort of put-call parity between the junior and the senior tranche:

$$L_a + Call(L, L_a) = Put(L, L_a) \quad (6)$$

If *f(L)* denotes the loss distribution, the pricing formula for the junior tranche looks like:

$$\Pi(0,a) = L_a - P(0,a)$$
where:
$$P(0,a) = e^{-rT}\left\{\int_0^{L_a} L\cdot f(L)dL + L_a\left[1-\int_0^{L_a} f(L)dL\right]\right\}$$
(7)

Note that we are assuming that the losses are allocated to the protection seller only at the maturity of the credit derivative.

Further manipulating eq. (3), we can also understand why the junior tranche is also called the "equity" tranche in the CDO jargon. Let's suppose that 100 is the nominal value of the underlying portfolio, and define by *V* the effective portfolio value at maturity (i.e. *V=100-L*) and by *D* the difference (100- L_a). Substituting into (3) these new definitions, we obtain:

$$\begin{aligned}\Pi(0,a) &= 100 - D - \min\{100 - D, 100 - V\} \\ &= 100 - D + \max\{D - 100, V - 100\} \\ &= 100 - D + \max\{D, V\} - 100 \\ &= \max\{0, V - D\}\end{aligned}$$
(8)

This can be seen as the expression of the equity of a company whose debt is equal to *D*. This analogy can be used to better understand the relationship between the holders of the different CDO tranches.

Let’s consider for example the case of a CDO whose underlying portfolio is not static but dynamically managed (i.e. there are regular replenishment of new credits in the basket according to some predetermined rules). The junior tranche is, in general, retained by the “manager”. If it is in the interest of the manager to keep as low as possible the expected default rate, there might be also an interest in increasing the volatility of the expected default rate. Whereas the first goal is shared by the holders of the senior tranche, the second is against their interests.

This explains why the replenishment rules are now designed to keep constant not only the expected default rate of the portfolio (by asking to keep constant its average rating), but also the way in which this goal is reached (by limiting the risk concentration by sectors, industries, ... to predetermined values).

Using the new notation, we can rewrite also the pay-out of the senior tranche:

$$\begin{aligned}\Pi(a,1) &= \min\{D, V\} \\ &= D - \max(0, D - V)\end{aligned} \quad (9)$$

Then, the senior tranche value is equal to the value of a portfolio composed of a zero-coupon bond (equal to *D*) and a short put option on *V* (with strike equal to *D*).

In general, the tranches over the equity are divided in at least two categories: “mezzanine” and “senior” (and, more recently, we have a fourth category, the “super senior”). The most senior tranche pricing formula has already been described in eq. (9).

Let's then evaluate an intermediate tranche, defined over the (α,β) percentile range:

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$$\begin{aligned}\Pi(a,b) &= \max\{\min\{L_b - L_a, L_b - L\}, 0\} \\ &= \max\{(L_b - L_a) + \min\{0, L_a - L\}, 0\} \\ &= (L_b - L_a) + \max\{\min\{0, L_a - L\}, -(L_b - L_a)\} \\ &= (L_b - L_a) + \min\{0, L_a - L\} + \max\{0, -(L_b - L_a) - \min\{0, L_a - L\}\} \\ &= (L_b - L_a) - \max\{0, L - L_a\} + \max\{0, -(L_b - L_a) - \min\{0, L_a - L\}\}\end{aligned} \quad (10)$$

In order to proceed with the manipulation of eq. (10), we have to note that:

$$(L_a - L_b) - \min\{0, L_a - L\} = e > 0 \text{ if } L = L_b + e \, . \quad (11)$$

Then, we can write $\max\{0, -(L_b - L_a) - \min\{0, L_a - L\}\} = \max\{0, L - L_b\}$. Coming back to eq (10), we have:

$$\begin{aligned}\Pi(a,b) &= (L_b - L_a) - \max\{0, L - L_a\} + \max\{0, L - L_b\} \\ &= (L_b - L_a) - P(a,b) \\ &\textit{where}: \\ P(a,b) &= \max\{0, L - L_a\} - \max\{0, L - L_b\}\end{aligned} \quad (12)$$

So, we can interpret the mezzanine tranche as a portfolio composed of a zero coupon bond and two call options, one short and the other long, on the portfolio's losses. The short position is the one which is always more in the-money and as such dominates in absolute value and in

Delta the long one. Then, a reduction in the expected default rate increases the value of the mezzanine tranche.

An increase in the volatility of the default rate increases the value of both options, and the overall effect is uncertain depending on the relative positioning of the expected loss rate with respect to the two bounds defining the percentile range. In normal cases, in which the expected loss rate is nearer to the lower bound of the range, the effect is negative, since the Vega of the short position tends to dominate the Vega of the long position. If, however, the expected loss rate increase so much to make the mezzanine tranche to look more like the equity one, then an increase in volatility can cause an increase in the value of the intermediate tranche.

2. The basic statistical framework

In principle, there are different ways to price a basket derivative. However, we will follow the so-called “reduced” form approach, as exposed by Duffie (1998).

2.1 The univariate setting

The idea behind the Duffie's approach is that the credit event can be (approximately)[2] modelled as a Poisson process, with intensity rate (or hazard rate) *h* depending on the length of the time interval. This implies that the probability of observing a credit event between time 0 and time *t* is equal to *h*t*. The general form of the Poisson distribution is, for *x=0,1,2, …*:

$$Poisson(x;h) = \frac{e^{-h\cdot t}(h\cdot t)^x}{x!} \qquad (13)$$

For a single firm, we are interested only in the case *x=0* , i.e. the firm does not default, whereas for *x>0*, the firm defaults. In our case, then, the probability that a company defaults before time *t* (the so-called "waiting time" or "hitting time" or, if reversed, "survival time") is Exponentially distributed:

$$Q(t) = 1 - e^{-h\cdot t} \qquad (14)$$

One can now easily understand why this approach to default modelling is taking ground, at least in the academic literature. If one consider the case of a continuously random hazard rate, the default probability can be written as:

$$Q(t) = 1 - E\left\{ e^{-\int_0^T h(t)\,dt} \right\} \qquad (15)$$

The expression on the rhs of eq (15) is very similar to the one of Zero Coupon Bond (ZCB). So it is sufficient to specify for *h* one the stochastic model described in the interest rate literature (for example, the Cox, Ingersoll, Ross square root process) and we have already available an analytical solution to eq. (15). However, we won't follow this approach and we will remain in the simple case of constant hazard rate.

Differentiating (14) with respect to time we obtain the density function expression:

$$q(t) = he^{-h \cdot t} \qquad (16)$$

The parameter *h* measures the (inverse of) expected life. So if *h=0.1*, we expect that the credit event will happen in 10 years time. For further digressions about the interpretation of the hazard rate, see for example Li (1999a).

Another nice property of Poisson processes with hazard rate *h*, is that the time until we observe *x* happenings is distributed as a Gamma function with parameters (*h*,*x*).

The functional form of the Gamma density function is:

$$q(t,T;x) = \frac{h^x}{\Gamma(x)}(T-t)^{x-1}e^{-h(T-t)} \qquad (17)$$

If *x*=1, we come back to the Exponential case.

2.2 The calibration of the univariate setting to market data

As far as the calibration of the model to the real world data, it is quite easy to find an expression for *h* which is proportional to the market spread paid on a corporate bond. More precisely, following the suggestion of JP Morgan (1999) we can state that the default probability over a certain time horizon, *T*, implied in the market spread, s, given a specific loss rate, *l*, is approximately equal to:

$$1-\frac{1}{\left(1+\frac{s}{l}\right)^T}$$
(18)

Since the probability of default in our Poisson modelling is explicitly defined in terms of *h*, we can obtain:

$$h=\ln\left(1+\frac{s}{l}\right)$$
(19)

A further approximation (Taylor series expansion of the log) allows us to state $h \approx s/l$.

2.3 The multivariate setting

One of the nice properties of the Poisson processes is that, if they are independent, their Survival Time joint distribution is still Exponential, with an hazard rate simply equal to the sum of the single hazard rates. In other words, if $\{t_1, t_2, ..., t_N\}$ is a vector of N independent Exponentially distributed survival times with associated hazard rates $\{h_1, h_2, ..., h_N\}$, their joint-distribution over a common time horizon is Exponential with hazard rate equal to $H = h_1 + h_2 + ... + h_N$.

The problem with the Exponential distribution arises when we want to model the correlation between the variables. In this case, in fact, it does not exist a unique functional specification for the multivariate Exponential.

We will use the functional form explored by Marshall and Olkin (1967), that has been used in its bivariate version by Li (1999 b) and it is behind the approach of Duffie-Singleton (1999).

In the case of just two risks, the bivariate Exponential distribution for the survival times can be expressed as:

$$S(T_1,T_2)=\exp\{-h_1T_1-h_2T_2-h_{12}\max(T_1,T_2)\}$$
(20)

The marginal survival probability function will be equal to:

$$S(T_1) = \exp\left\{-\hat{h}_1 \cdot T_1\right\}$$
$$= \exp\left\{-(h_1 + h_{12})T_1\right\}$$
(21)

If the time horizon is common, such as in our case, we have:

$$S(T,T) = \exp\{-H \cdot T\}$$
$$= \exp\{-(h_1 + h_2 + h_{12})T\}$$
(22)

So that the *hazard rate* for the entire portfolio becomes equal to: $H = h_1 + h_2 + h_{12}$. Note that if we want to express the hazard rate of the portfolio in terms of the marginal hazard rates, we would have:

$$S(T,T) = \exp\{-H \cdot T\}$$
$$= \exp\left\{-(\hat{h}_1 + \hat{h}_2 - h_{12})T\right\}$$
(23)

The correlation between the surviv al times is equal to:

$$r_{12} = \frac{h_{12}}{(h_1 + h_2 + h_{12})}$$
$$= \frac{h_{12}}{(\hat{h}_1 + \hat{h}_2 - h_{12})}$$
(24)

Let's consider now the 3-variate case. We define:

$$S(T_1,T_2,T_3) = \exp\{-h_1T_1 - h_2T_2 - h_3T_3 - h_{12}\max(T_1,T_2)$$
$$-h_{13}\max(T_1,T_3) - h_{23}\max(T_2,T_3) - h_{123}\max(T_1,T_2,T_3)\}$$
(25)

The (marginal) joint-distribution for the first two variables is:

$$S(T_1,T_2) = \exp\{-(h_1 + h_{13})T_1 - (h_2 + h_{23})T_2 - (h_{12} + h_{123})\max(T_1,T_2)\ \}$$
(26)

The marginal distribution of the first variable is:

$$S(T_1) = \exp\left\{-\hat{h}_1 \cdot T_1\right\} = \exp\left\{-(h_1 + h_{12} + h_{13} + h_{123})T_1\right\} \quad (27)$$

So, analogously with the previous section, we have:

$$r_{12} = \frac{h_{12} + h_{123}}{(h_1 + h_{13}) + (h_2 + h_{23}) + (h_{12} + h_{123})} \quad (28)$$

In the general case of *n*-variables, we have that the marginal distribution of a single survival time is equal to:

$$S(T_1) = \exp\left\{-(h_1 + \sum_{i=2}^{n} h_{1i} + \sum_{i=3}^{n} h_{12i} + \ldots + h_{123\ldots n})T_1\right\} \quad (29)$$

We omit the joint distribution definition because its notation is too complex. The correlation between two variables will be be obtained in a similar way as in the bivariate or tri-variate case.

2.4 A simplified multivariate setting

The real problem with the multivariate exponential distribution devised by Marshall-Olkin is the parameter estimation. If we specify only the first two moments of the distribution (i.e. expected default rate and correlation among them) we are unable to obtain a unique solution for the parameters' value, except for the bivariate case.

In the model of paragraph 2.3, for example, a 3-variables correlated system is generated by 7 uncorrelated shocks: 3 shocks are idiosyncratic to the 3 variables, 1 shock is common to all 3 and the remaining 3 are instead specific to each of the couplets.

So, we can simplify the model by eliminating the occurrence of shocks specific to the 3 couplets. The 3 variables are still correlated among themselves, but only through one common process governing the occurrences of simultaneous shocks to the pairs of variables. This will limit the value of the correlation, as we will see.

Using our simplified model, we can identify the system by putting $h_{12} = h_{13} = h_{23} = 0$ in the 3 variables case. This reduces the values that the correlation coefficient between any two variables can assume. If the time horizon is common, such as in our case, we have:

$$S(T) = \exp\{-(h_1 + h_2 + h_3 + h_{123})T\} \qquad (30)$$

In the general multivariate case, we will have:

$$S(T) = \exp\left\{-\left(\sum_{i=1}^{n} h_i + h_{12..n}\right)T\right\} \qquad (31)$$

The *hazard rate* for the entire portfolio becomes equal to the sum inside the brackets:

$$H = \sum_{i=1}^{n} h_i + h_{12...n} \qquad (32)$$

Since we know the hazard rate for the entire portfolio, we can calibrate the parameters of the model to our laboratory data (expected default rates and correlation) and simply apply the formulas developed for the univariate case or for the default independence case to price the basket derivatives.

We will now consider a special case that can be useful for testing particular pricing models. We want to find an expression for h_i and $h_{i,j}$ under the special case of a basket of two names of identical credit quality and with the same correlation among themselves. This means that their hazard rates are identical:

$$\hat{h}_1 = \hat{h}_2 = \hat{h} \qquad (33)$$

and without any loss of generality, we can assume also that:

$$h_1 = h_2 = h \qquad (34)$$

Let's assume that the correlation between any two names is equal to ρ. For example, it is often assumed that the correlation coefficient is equal to 30% inside the same industry.

Solving eq. (24) for h_{12}, we have

$$h_{12} = (h_1 + h_2)\frac{r_{12}}{(1-r_{12})}$$
$$= 2h\frac{r}{(1-r)}$$

(35)

We can then obtain:

$$h = \hat{h}\frac{1-r}{1+r}$$
$$h_{1,2} = 2\hat{h}\frac{r}{1+r}$$
$$H = 2\hat{h}\frac{1}{1+r}$$

(36)

The simplified 2-variables model can easily be extended to *N* variables with identical hazard rates and correlation among themselves. In particular, if we define the marginal hazard rate for the *i*-th name as:

$$\hat{h} = h + h_{12...N}$$

(37)

we can find that:

$$h = \hat{h}\frac{1-r}{1+r}$$
$$h_{12...N} = 2\hat{h}\frac{r}{1+r}$$
$$H = \hat{h}\left[N - (N-1)2\frac{r}{1+r}\right]$$

(38)

2.5 A note on simulating default correlation with Copulas

In order to price basket credit derivatives, it has become a common practice to use Monte Carlo simulations where the survival times are randomly generated using a Normal distribution and the associated Copula function (see Li, 1999). In the following, we want to see what are the implications of this hypothesis.

In practice, if we limit ourselves to the bivariate case, the Normal Copula function is given by:

$$C(T_1, T_2) = NBiv\left\{N^{-1}\{F(T_1)\}, N^{-1}\{F(T_2)\}, \rho\right\} \tag{39}$$

where *F(.)* denotes the marginal distribution (in our case the Exponential), *N(.)* denotes the univariate standardised Normal distribution and *NBiv(.)* the bivariate Normal distribution with correlation equal to ρ. So ρ expresses the hypothesis (or estimate) about the correlation between the survival times of the two assets underlying our basket derivative.

A simpler Copula function (which is however more difficult to apply in the multivariate case) is the following one (see Li, 1999) for ρ>0:

$$C(T_1, T_2) = (1-\rho)F(T_1)F(T_2) + \rho \min\{F(T_1); F(T_2)\} \tag{40}$$

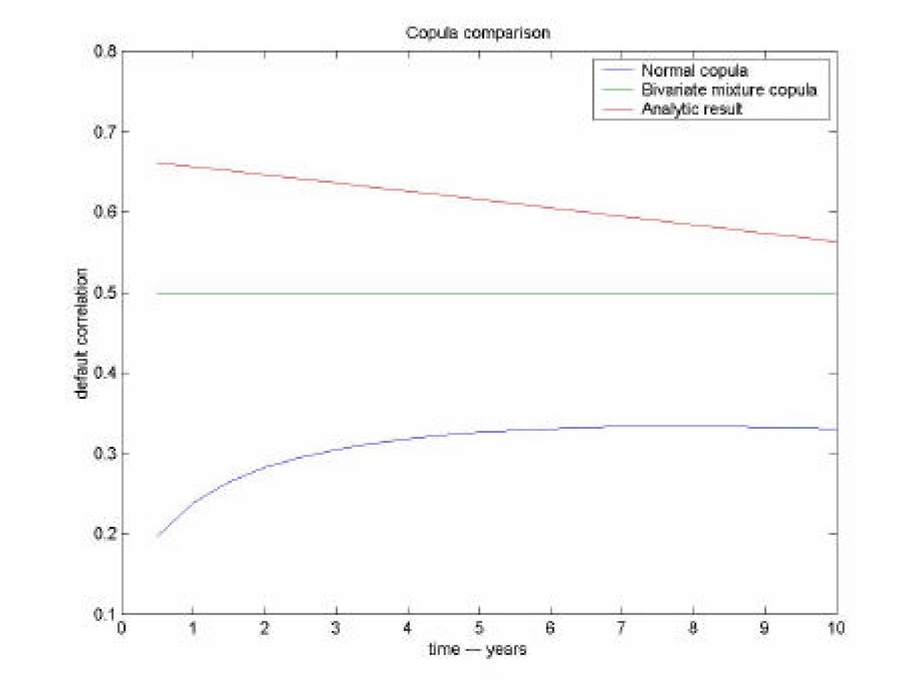
In the figure below, we compare the value of the default's correlation implied by the survival times correlation that we are assuming both in the analytical Multivariate exponential distribution, see eq. (24) and in the joint-distribution obtained via Copula functions. In order to obtain a common value for the survival times' correlation, i.e. ρ=0.5, we assumed a value of 0.09 for h_1, h_2 and we solved eq. (24) in order to obtain h_{12}.

Note that the default correlation is not equal to survival times correlation. In fact, the default event is a binomial phenomenon. Using the notation of eq. (10), if *Q* is the probability that an asset defaults (over the unit interval), its variance will be equal to *Q*[1-Q]* and the correlation between asset 1 and 2 defaults will be equal to:

$$\rho_{1,2}(default) = \frac{Q_{1,2} - Q_1 Q_2}{\sqrt{Q_1[1-Q_1]Q_2[1-Q_2]}} \tag{41}$$

where $Q_{1,2}$ can be obtained analytically or by simulation, through the Copula function approach.

Fig. 1 - Implied default correlation

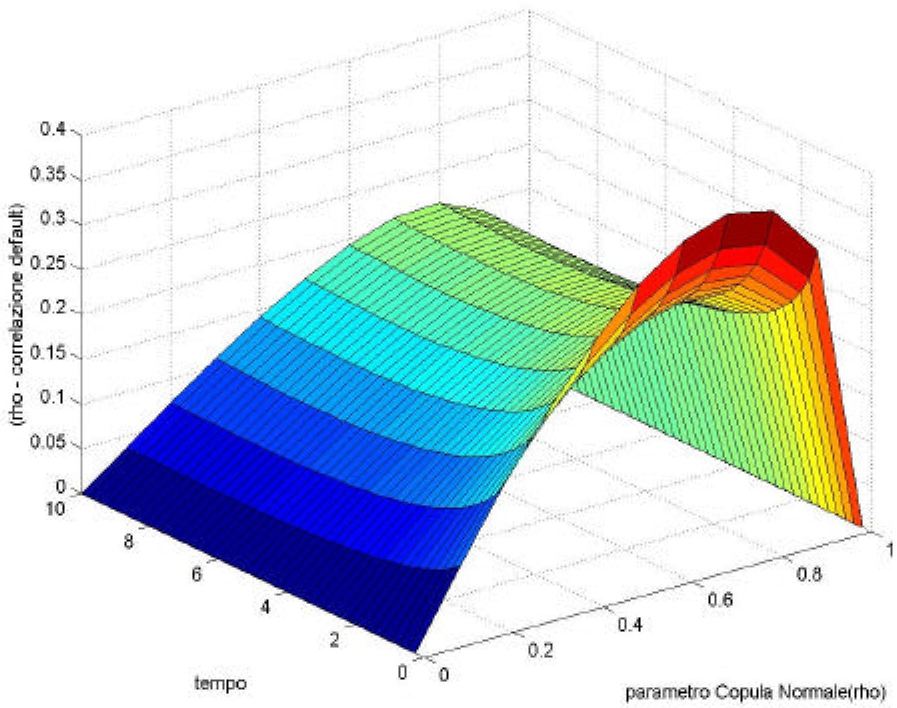


The Normal copula produces the lower line, the Simple copula the intermediate line, the analytic distribution the upper line. As one can see, it is sufficient to change the functional form that from the marginal distributions of the survival times brings to the joint distribution and we obtain a very different value for the defaults correlation.

It has also to be noted that default correlation varies according to time in the Normal copula case. This result is not unexpected since the Pearson correlation coefficient depends on the marginal distribution (see Frees-Valdez, 1998): in our case, the default probability increases as time goes by. The default correlation changes, then, also if we leave unchanged the survival times correlation.

These considerations should then be seen as a note of alert against an "easy" use of the Normal copula function in Monte Carlo pricing of basket derivatives. In fig. 2 we showed the difference between the ρ parameter and the implied default correlation, as time and ρ itself vary. We only left the two hazard rates h_1, h_2 unchanged and equal to 0.09.

Fig 2. Difference between Survival Times correlation (ρ) and implied default correlation



3. Pricing basket derivatives

The loss severity in case of default depends on the principal outstanding and the loss rate or both. For the sake of simplicity, we will carry on assuming that the loss rate is constant and identical for all the assets. This is not so unusual if the assets underlying a CDO are of the same type (example: bonds, leveraged loans, mortgages, ...). We assume that the only things that can change across the assets are (i) the hazard rates and (ii) the principal outstanding.

3.1 Pricing first-to-default credit derivatives

Let's imagine a basket of N corporate bonds of identical nominal value, L. Let's imagine also that these corporate bonds have a different credit risk, as represented by the hazard rates, and that their maturity is greater than or equal to the one of the credit derivative, T.

If we are pricing a first-to-default derivative on this basket, the payout of the contract is equal to L in case one or more of the names in the basket default before T. We assume that the contract ends when the event occurs.

For the sake of simplicity, let's assume a constant and deterministic risk-free term structure. The price/cost of the first-to-default contract is equal to (Li, 1999a):

$$P = L\int_0^T e^{-rt} He^{-H\cdot t} dt$$
$$= L\frac{H}{r+H}\left(1-e^{-T(r+H)}\right)$$
(42)

Where $H = h_1 + h_2 + ... + h_N$ in case of default independence or it is equal to the more complicated expression depicted in eq. (32) in case of default dependence.

The analytical solution allows us to better understand the dynamics of the first-to-default price. For example, the derivative of P with respect to H is positive. This implies that a worsening of the credit quality causes an increase in the cost of the contract. Since those contracts are quoted in terms of spread over Libor, this implies an increase in the required spread.

The effect of a change in the correlation between the survival times is instead a bit more difficult to find out. First of all, we note from eq. (35) that an increase in ρ causes an increase in h_{12}. But now we have to be careful. There is no reason why an increase in ρ should affect the credit quality of the single names included in the basket. So, if we leave $\hat{h}_i$ unchanged (for each *i*-th name), we can apply eq. (23) and observe that an increase in h_{12} causes a reduction in *H*. This in turn implies that the cost of the first-to-default (or the spread) decreases!

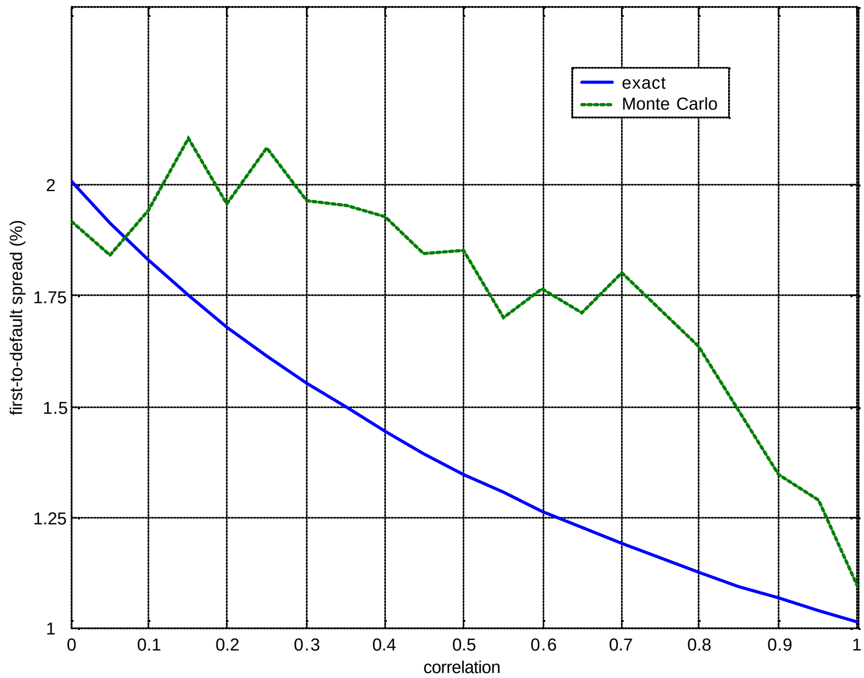
This result is at first sight counterintuitive. We are used to the idea that “diversification” among the asset is “good”. For the first-to-default, this is not the case. Why?

Perfect correlation means that, if one of the names defaults, all the others default, too. Since the first-to-default’s pay-out is fixed and independent of the defaults’ number, it is perfectly analogous to buy the single bond or the first-to-default basket. The spread on the first-to-default should be identical to the spread of one of the bonds.

If instead the bonds have zero correlation among themselves, the probability of two or more names defaulting contemporaneously is very low. But there are many more States of Nature where at least one of the bonds defaults. The probability of observing one default is equal to *N* times the probability of one of the names defaulting. Since the spread paid by each bond reflects the default probability, the spread paid by the first-to-default will be equal to the sum of the spreads paid by the bonds included in the basket.

In the figure below we have calculated the price of a first-to-default credit derivative written on two names.

Fig 3- First-to-default spread



In order to appreciate the usefulness of the analytical formula, we plot in Fig. 3 also the pricing schedule obtained via a simple Monte Carlo simulation[3]. Apart from the irregular shape of the latter, we can notice a difference in the curvature. In principle, since the analytical result has been obtained with one of the possible specification of the multivariate exponential distribution, it is not possible to conclude that simulating we obtain a result worse than the analytical one. However, we will see in a moment that the curvature of the analytical schedule is more appropriate.

Consider, in fact, the case of negative correlation. According to our analytical model, this implies that the spread increases. In the limit, when correlation is perfectly negative, i.e. equal to –1, the spread goes to infinity, coherently with the curvature observed in the more realistic (0,1) interval. Is this correct? The answer is yes. If the correlation is equal to –1, this implies that the States of Nature are only two: company 1 survives and company 2 defaults, or viceversa. This might happen for example if the two companies are in Bertrand competition and have different production costs. In every State of Nature, then, one of the company defaults.

3.2 Dealing with heterogeneous loss severity

This case arises when the outstanding principals or the recovery rates differ significantly across the single assets composing the underlying portfolio. As we have already said, we will consider as the only cause of diversified loss severity a diversified principal outstanding.

Let's imagine that the principal outstanding varies. Since the recovery rate is assumed equal to zero, this implies that the loss associated to the i-th name defaulting, L_i, will be equal to the principal of the associated bond that has been included in the basket. The price will be equal to:

$$P = \sum_{i=1}^{N} L_i \left[\int_0^t f(i,t) e^{-rt} H e^{-H \cdot t} dt \right]$$

(43)

where *f(i,t)* denotes the probability that at time *t* the first to default is the i-th name, given that a default occurred. This probability is equal to the ratio between the probability that the i-th name defaults ($h_i t$) and the probability that at least one name default (Ht):

$$f(i,t) = \frac{h_i}{H}$$

(44)

Substituting back, we obtain:

$$P \approx \sum_{i=1}^{N} h_i L_i \left[\int_0^t e^{-[r+H]t} dt \right] = \sum_{i=1}^{N} h_i L_i \left[\frac{1}{r+H} \left(1 - e^{-[r+H]t}\right) \right]$$

(45)

Now, we can study the effect of a concentrated or dispersed distribution for the losses associated to the single names. For the sake of simplicity, let's imagine that the risk-free rate is equal to zero. The pricing formula becomes:

$$P \approx \left(1 - e^{-H \cdot t}\right) \sum_{i=1}^{N} L_i \frac{h_i}{H}$$

(46)

We could have obtained a similar result if we had considered a first-to-default contract written on *N* bonds with identical outstanding principals, all equals to the weighted average of the outstanding principals of the bonds included in our "real" portfolio. The weights to be used in the average are given by the ratio $\frac{h_i}{H}$.

Note that the simple arithmetic average of the principal outstanding is the correct choice only if the names included in the portfolio are of similar credit quality, i.e. $h_i = h$. It is also interesting to note a sort of "duality" property: in absence of default correlation, the risk exposures and hazard rates dispersions are important only if they come together.

3.2 Pricing first-x-to-default

Consider now the case of pricing a derivative that covers the first *M* names defaulting. If the single hazard rates are small and quite homogeneous and *M* is small compared to the number of names included in the basket, we can use the Gamma density function and the pricing equation will be:

$$\begin{aligned} P(1,2,...,M) &= L\sum_{x=1}^{M} P(x) \\ &= L\sum_{x=1}^{M}\int_0^T e^{-rt}\frac{H^x}{\Gamma(x)}t^{x-1}e^{-H\cdot t}dt \end{aligned} \tag{47}$$

Where $\Gamma(.)$ denotes the gamma function. Clearly, this sum of integrals can be solved by numerical integration, but for the moment we are interested in a closed-form solution.

We can interpret the single terms of the summation as the "truncated" moment generating function of the Gamma density, with coefficient equal to *–r*. Denoting the moment generating function of the Gamma density with $\Im$, we can manipulate the single terms of the r.h.s of eq. (47) in the following way:

$$\begin{aligned} P(x) &= \left[\int_0^\infty e^{-rt}\frac{H^x}{\Gamma(x)}(t)^{x-1}e^{-Ht}dt - \int_T^\infty e^{-rt}\frac{H^x}{\Gamma(x)}(t)^{x-1}e^{-Ht}dt\right] \\ &= \left[\Im(-r) - e^{-(r+H)T}\int_0^\infty e^{-rt}\frac{H^x}{\Gamma(x)}(T+t)^{x-1}e^{-Ht}dt\right] \end{aligned} \tag{48}$$

where $\Im$ (t) is equal to (see Mood, Graybill, Boes, 1974):

$$\Im(t) = \left(\frac{H}{H-t}\right)^x \tag{49}$$

We can solve the integral expression in eq. (48) substituting in the functional form of $\Im$. For example, for *x*=2, recalling that $\Gamma(2)=2$, we have:

$$P(x)=\left(\frac{H}{H+r}\right)^2\left(1-e^{-(r+H)T}\right)-e^{-(r+H)T}T\frac{H^2}{\Gamma(2)}\int_0^\infty e^{-rt}e^{-Ht}dt$$
$$=\left(\frac{H}{H+r}\right)^2\left(1-e^{-(r+H)T}\right)-e^{-(r+H)T}T\frac{H^2}{2(H+r)}$$
(50)

For *x*=*3*, we have:

$$P(x)=\left(\frac{H}{H+r}\right)^3\left(1-e^{-(r+H)T}\right)-e^{-(r+H)T}T^2\frac{H^3}{\Gamma(3)}\int_0^\infty e^{-rt}e^{-Ht}dt$$
$$-e^{-(r+H)T}(2T)\int_0^\infty e^{-rt}\frac{H^3}{\Gamma(3)}te^{-Ht}dt$$
(51)

The last integral on the r.h.s. of the equation can be rewritten in the following way:

$$\int_0^\infty e^{-rt}\frac{H^3}{G(3)}te^{-Ht}dt=\frac{H}{3}\int_0^\infty e^{-rt}\frac{H^2}{G(2)}te^{-Ht}dt$$
$$=\frac{H}{3}\left(\frac{H}{H+r}\right)^2$$
(52)

Substituting back, and recalling that $\Gamma(3)=6$:

$$P(x)=\left(\frac{H}{H+r}\right)^3\left(1-e^{-(r+H)T}\right)-e^{-(r+H)T}T^2\frac{H^3}{6}\frac{1}{H+r}$$
$$-e^{-(r+H)T}(2T)\frac{H}{3}\left(\frac{H}{H+r}\right)^2$$
(53)

A closed form solution can be found also for higher values of *x*, by replicating the decomposition scheme depicted above. However, the final expression and the calculations become soon quite cumbersome.

In order to obtain a further simplification, we can approximate the solution by assuming that the contract pays out only at maturity. In this case, in fact, we can apply the Poisson probability formulas:

$$P*(n) = Le^{-r(T-t)} \sum_{i=1}^{n} i \cdot \Pr ob(n.def = i)$$
$$= Le^{-r(T-t)} \sum_{i=1}^{n} i \cdot \frac{e^{-h\cdot(T-t)} (h \cdot (T-T))^i}{i!}$$
(54)

Apart from simplicity, eq (54) is very useful because it allows to show the approximation that we are incurring when using the Poisson processes to model companies' default. In fact, the Poisson hypothesis implies a positive probability of observing a number of failures bigger than the number of names in the basket! However, if *n* is sufficiently large with respect to the number of defaults covered by the basket derivative and the probability of default sufficiently small, the approximation error is negligible.

3.3 Pricing percentile basket-derivatives

In order to price that kind of securities the standard practice has become the one based on the so-called "diversity score". Let's imagine to have a portfolio of *N* names whose default intensity is correlated and whose bonds constitute the collateral pool of a CDO. We want to find an idealized comparison portfolio composed of *N** independent names (where *N** is less than or equal to *N*), with identical default probability. This default probability is supposed equal the weighted average default probability of the names of the actual portfolio. The actuale and the idealized portfolio have the same total face value. The face value of the single (identical) names of the idealized portfolio will then be equal to *1/N** times the total face value.

If we can find this comparison portfolio, *N** is called the diversity score of the actual portfolio and the pricing of the CDO can be extremely simplified. If we have *N** identical names with default probability equal to *Q*, as in eq. (14), the probability of observing *M* defaults can be calculated using the binomial formula:

$$\frac{N*!}{(N*-M)!M!} Q^M (1-Q)^{N*-M}$$
(55)

Obviously, the difficult task is to find out the diversity score. Moody's, for example, tabulates the diversity score for portfolios of companies belonging to the same industrial sector. The underlying hypothesis is that there is positive correlation between the defaults of firms belonging to the same industrial sector.

The Moody's methodology is proprietary and there is no disclosure about the underlying statistical model. However, if we use our simplified multivariate setting, we can gain some insights in Moody's calculations. We said that the idealized portfolio's names need to have the same default probability of the names included in the actual portfolio. This means that the hazard rate is identical. We could then say that the idealised portfolio and the actual portfolio are similar if their aggregate hazard rates are identical. Recalling that the hazard rate for the idealized portfolio is $M \cdot \hat{h}$ and that the hazard rate for the actual portfolio is given by eq. (38), we can find a simple expression for the diversity score:

$$M = N - (N-1)2\frac{r}{1+r} \qquad (56)$$

With eq. (56) we can calculate the implied correlation coefficient of Moody's diversity score.

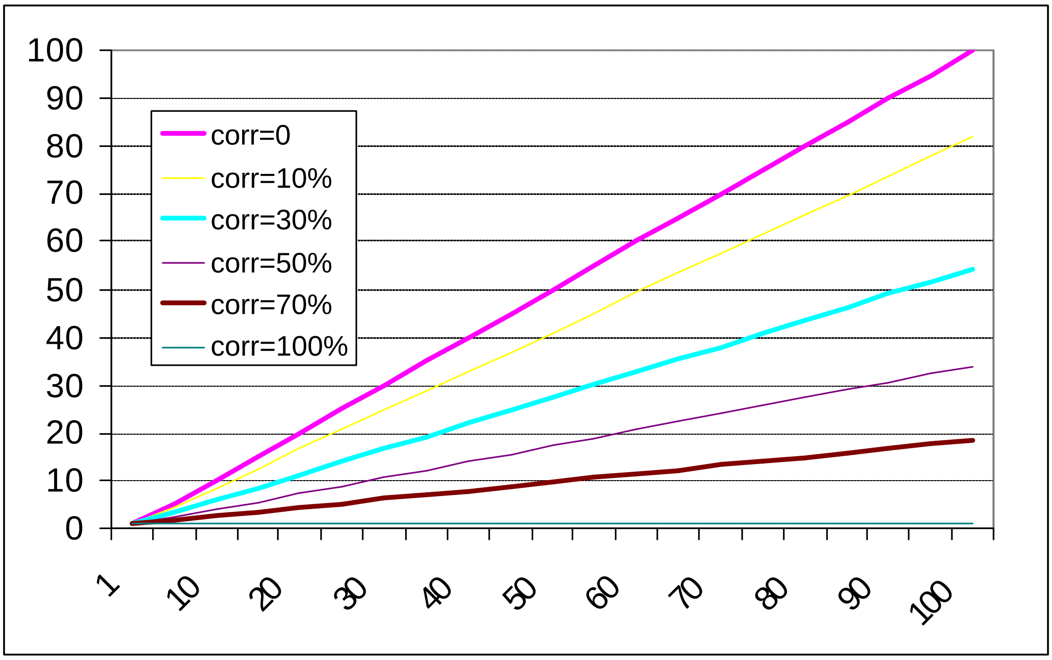
Tab. 1 – Moody's diversity score and implied correlation

Firms in industry*	Diversity score*	Implied Correlation**
1	1	---
2	1.5	33.3%
3	2	33.3%
4	2.33	38.6%
5	2.67	41.1%
6	3	42.9%
7	3.25	45.5%
8	3.5	47.4%
9	3.75	48.8%
10	4	50.0%
>10	evaluated on a case-by-case basis	---

* source: Moody's; ** estimated via eq (56)

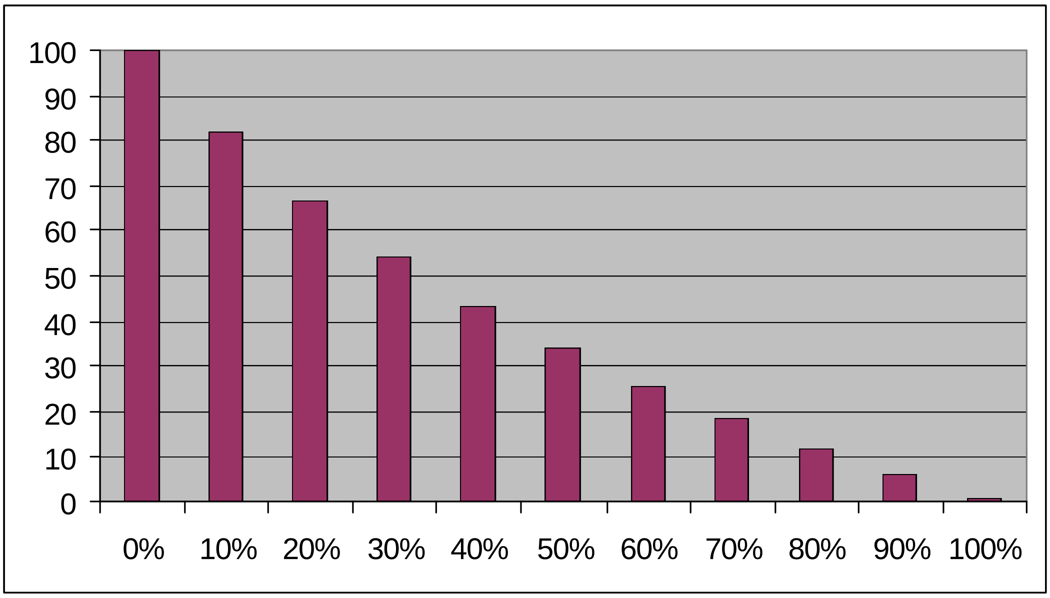
Moody's does not tabulate the diversity scores for a number of firms higher than 10. In fig. 4, we show the diversity score for higher values of *N* and for different values of the correlation coefficient, calculated using our simple eq. (56).

Fig 4- Diversity Score Lines



Finally, we show in Fig.5 the diversity scores for a portfolio of 100 names, under different hypotheses about the value of the correlation among the default intensities.

Fig. 5- Diversity Scores for N=100 and varying correlation



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Notes

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[1] This paper has been presented at 2nd International Conference on "Managing Credit and Market Risk", Università di Verona, June 7-8, 2001

[2] The values that a Poisson random variable can take are all the positive integers, whereas we are interested only in the two events: fails (0) or not fails (1).

[3] Note that the simulations are obtained via the standard approach of extracting random values from a multivariate Normal distribution, characterised by uniform correlation, ρ. For every number extracted, we calculate the corresponding cumulative probability, according to the univariate standardised Normal distribution. This procedure generates a set of correlated random numbers that are distributed between 0 and 1, and that can be interpreted as the survival probabilities. Now, we can invert the survival probabilities to obtain the survival time and calculate the number of times that the survival time is less than the maturity of the contract.