LIUC eBook

Financial Modelling and Management Part I

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LIUC Università Cattaneo

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Introduction

Orandum est ut sit mens sana in corpore sano

Decimus Iunius Iuvenalis, I-II century AD

Financial Modelling and Management – Part I concerns the application of scientific tools to **investment** and **funding** activities that are represented in terms of (**semi**)**deterministic** cash flows. More precisely, it deals with some **validated** models and procedures that support decisions on

- the comparison of banking and near banking transactions, e.g. a time deposit, a home mortgage loan, a security lending arrangement, a lease of vehicles, plant, or machinery;
- the appraisal of real investment projects;
- the management of bond portfolios.

The presentation is both **qualitative** and **quantitative**. Emphasis is placed on **data** and **procedures** that are used or taken into consideration in **business practice**. As a consequence, students will learn which data to analyse and how to go about **financial analysis**, especially in connection with bilateral loans and bonds. Students will also understand in depth

- how a **mortgage loan** application is managed and analysed by an Italian bank;
- how a term structure of the **money market** is estimated in the EU;
- how the **actual yield** on a fixed income security is linked to the **yield to maturity**;
- how creditworthiness is rated by **international agencies**. In doing so, the **empirical evidence** on the performance of US corporate bonds is outlined.

This work is arranged in 5 sections of increasing complexity, with all subsequent sections expanding on the first one. All the relevant aspects of a financial problem are taken into account through a **problem-oriented** and hence multidisciplinary approach. Therefore,

- reference is made to the theoretical principles and practical notions of other related subjects such as **business economics, industrial economics, financial economics**, and **applied statistics**;
- theoretical notions are briefly explained and illustrated by examples that are consistent with business practice. Emphasis is placed, whenever possible, on operational procedures that are in line with the Italian law;
- exercises are based on real or realistic data.

The authors share the belief with other Italian colleagues that a fruitful theory rests on a solid and practical basis, and vice versa. Accordingly, learning and retaining **Financial Modelling and Management – Part I** should be made easier by a twofold course of reading

- the theoretical one, concerned with analytical processes or statistical enquiries and their peculiarities;
- the operational one, focused on financial contracts, financial transactions, and business practice.

This work has resulted from the teaching of the latter author. The fine tuning to business practice is due to the former author. It goes along with Financial Modelling and Management – Part II, reserved to LIUC students and unpublished yet.

Bergamo–Milano, 28 June 2018

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1. Basics of financial mathematics

1.1. Accumulation and discount of amounts of money

Let time t be measured in **years** and 0 be the present time. Suppose that an amount of money C is lent over the term $[0;t]$ and is repaid with a lump sum. Since lending is rewarded, a larger **future value** or **accumulation** $FV = C + I$ will be paid from the borrower to the lender at time t in return for the loan of the **principal**, or **capital**, C over the term $[0;t]$. The difference I is the accrued **interest**, i.e. the lender's reward, which *coeteris paribus* (all other things being equal) is larger, the more distant the **maturity** t is. As we have

$$FV = Cf(t) \text{ with } f(0) = 1 \text{ and } \frac{df(t)}{dt} > 0 \text{ for all } t \geq 0$$

where $f(t)$ is an **accumulation factor**, the **accumulation of money** is a process whereby interest accrues as time goes by

capital C	$FV = C + I = Cf(t)$
time 0	**time** t since start

Interest rate and **credit risk** are assumed away for the time being. In other words, we disregard that the expectations implicit in the **term structure** of the money market at time 0 may subsequently be not met by the time patterns of interest rates, such as **Eonia**, **Euribor**, and swap rates, which are introduced in the sequel; moreover, the borrower is assumed to meet surely all his/her contractual obligations. Finally, **settlement lags**, **commissions**, **fees**, and **taxes** are not explicitly considered; a few exercises make exception to the rule. Our elementary theory of **annuities** is developed in a **deterministic** setting, where the amounts of money as well as the interest rates, both present and future, are supposed to be **known**.

Suppose that a credit with **face value** C due at the **maturity** t is sold at time 0 at a smaller **present value** $PV = C - D$, the difference D being the **discount**. The buyer becomes a lender: he or she will be paid the future value C in return for a smaller loan PV over the term $[0;t]$. As we have

$$PVf(t) = C \text{ and hence } PV = \frac{C}{f(t)} \text{ with } f(0) = 1 \text{ and } \frac{df(t)}{dt} > 0 \text{ for all } t \geq 0$$

where $\frac{1}{f(t)}$ is a conjugate **discount factor**, the **discount of money** is a process opposite to the previous one whereby an amount payable at a later date is reduced to a smaller amount payable earlier, which *coeteris paribus* is lower, the more distant the maturity t is

$PV = C - D = Cf(t)^{-1}$	**face value** C
time 0	**maturity**: time t

The future and present values are 2 **linear** operators in the amounts of money. Therefore, if 2 amounts of money C_1 and C_2 are lent over the terms $[t_1;t]$ and $[t_2;t]$ respectively, their accumulation at time t will be $FV = C_1 f(t-t_1) + C_2 f(t-t_2)$. Moreover, if 2 amounts of money C_1 and C_2 are payable at the times t_1 and t_2, their present value at time 0 will be $PV = C_1 f(t_1)^{-1} + C_2 f(t_2)^{-1}$.

To performs these financial calculations, an accumulation or a discount rule has to be set so that the accumulation factor $f(t)$ and the conjugate discount factor $f(t)^{-1}$ take an analytical form. Let i be a yearly rate of interest and d a yearly rate of commercial discount; in the following we will examine the 3 rules used in practice, namely that of

- **simple interest**, whereby $f(t) = (1+it)$;
- **compound interest**, whereby $f(t) = (1+i)^t$;
- **commercial discount**, whereby $f(t)^{-1} = (1-dt)$ for $t < \frac{1}{d}$.

In principle, the rules of **simple interest** and **commercial discount** should only be applied to **short-term** transactions, which last less than 12-18 months. Therefore, the rule of compound interest should be applied to **medium-** and **long-term** transactions; the former last between 12-18 months and 5 years, whereas the latter last more than 5 years.

When not stated otherwise, 30-day months are considered throughout this section, in line with the 30/360 European day count convention introduced in Example **1** along with the actual/360 and actual/365 day count conventions. As shown in Example **2**, a term of 1 year, 6 months, and 18 days is expressed as $t = 1 + \frac{6}{12} + \frac{18}{360} = 1{,}55$ years; the reverse calculation is performed in Exercise **1** and Exercise **7**.

Simple interest

Let time t be measured in **years**, 0 be the present time, and i be the **yearly** rate of simple interest, i.e. the yearly interest on a unit capital. Suppose that a principal, or capital, C is lent over the term $[0;t]$. Since **simple interest** increases linearly with time by the equation

$$I = Cit$$

the future value, or accumulation

$$FV = C + I = C + Cit = C(1 + it)$$

will be paid from the borrower to the lender at time t in return for the loan of C over the term $[0;t]$. Therefore, the future value of C at time t is equal to the principal C times the linear **accumulation factor** $f(t) = (1 + it)$.

Example 1.

€100.000 are lent from Wednesday, September 16th, to Wednesday, December 16th at the yearly rate of 1%. Find the simple interest by using the following day count conventions:

a) actual/360 or actual/365;

b) 30/360 European.

As the last (first) day must (not) be taken into account, the accumulation can be lent from Wednesday, December 16th.

Solution.

a) As the **actual** term lasts $14 + 31 + 30 + 16 = 91$ days, we have

$$I = Cit = 100.000 * 0{,}01 * \frac{91}{360} = 252{,}78\ €$$

$$I = Cit = 100.000 * 0{,}01 * \frac{91}{365} = 249{,}32\ €$$

b) The **conventional** term lasts $14 + 30 + 30 + 16 = 90$ days, since 30-day months are considered; moreover, should the initial or final date fall on the 31st of a month, it would be shifted to the 30st . Thus, we have

$$I = Cit = 100.000 * 0,01 * \frac{90}{360} = 250,00 \text{ €}$$

Example 2.

€25.000 are lent for 1 year, 6 months and 18 days at the yearly rate of 6%; find simple interest and accumulation under the assumption that each month has 30 days.

Solution.

We have

$$I = Cit = 25.000 * 0,06 * \left(1 + \frac{6}{12} + \frac{18}{360}\right) = 25.000 * 0,06 * 1,55 = 2.325 \text{ €}$$

$$FV = C + I = C(1 + it) = 25.000 + 2.325 = 25.000(1 + 0,06 * 1,55) = 27.325 \text{ €}$$

Example 3.

At the beginning of a certain year €5.000 are lent at the yearly rate of 4%, with €2.500 being lent after 9 more months. Find interest and accumulation after 12 more months.

Solution.

Recall that the future values of the 2 transactions can be summed, as the future value is a **linear** operator in the amounts of money. Therefore, I_A and I_B can be summed as well. We have

$$I = I_A + I_B = 5.000 * 0,04 * \frac{21}{12} + 2.500 * 0,04 = 450 \text{ €}$$

$$FV = (C_A + C_B) + I = (5.000 + 2.500) + 450 = 5.000 * 1,07 + 2.500 * 1,04 = 7.950 \text{ €}$$

REMARK.

Owing to the linearity of simple interest, its half-yearly amount $Ci0,5$ is half the yearly amount Ci, its quarterly amount $Ci0,25$ is a quarter of the yearly amount Ci, etc. The same proportions apply to the **equivalent** rates of simple interest, i.e. the periodic interests on a unit capital: the half-yearly rate is $i0,5$, the quarterly rate is $i0,25$, etc.

Example 4.

€50.000 are lent for 1 year and 3 months under the simple interest rule. The accumulation after 3 months is worth €50.500. Find **a**) the yearly accumulation; **b**) the final accumulation; **c**) the quarterly interest rate; **d**) the yearly interest rate *i*.

Solution.

Since the quarterly interest is $50.500 - 50.000 = 500$ €,

a) the yearly interest and the yearly accumulation are $500 * 4 = 2.000$ € and $50.000 + 2.000 = 52.000$ € respectively;

b) the final interest and accumulation are $500 * 5 = 2.500$ € and $50.000 + 2.500 = 52.500$ € respectively;

c) the quarterly interest rate is $500/50.000 = 1\%$;

d) the yearly interest rate is $i = 4 * 500/50.000 = 4 * 1\% = 4\%$, i.e. 4 times the quarterly rate.

Compound interest

Let time *t* be measured in **years**, 0 be the present time, and *i* be the rate of compound interest **effective per year**. Suppose that a principal, or capital, *C* is lent over the term $[0;t]$. If interest is **compounded yearly** according to the **exponential** convention, the future value, or accumulation, *FV* at time *t* of a principal *C* is

$$FV = C(1+i)^t$$

so that the amount $C(1+i)^t$ will paid from the borrower to the lender at time *t* in return for the loan of *C* over the term $[0;t]$. Therefore, the future value of *C* at time *t* is equal to the principal *C* times the exponential **accumulation factor** $f(t) = (1+i)^t$. The overall **compound interest** *I* at time *t* is

$$I = FV - C = C(1+i)^t - C = C\left[(1+i)^t - 1\right]$$

For $i = 5\%$, the compound interest on a unit principal $I = 1{,}05^t - 1$ is worth

$$\begin{array}{rl} 1{,}05 - 1 = 0{,}05000 & \text{after } \mathbf{1} \text{ year} \\ 1{,}1025 - 1 = 0{,}10250 & \text{after } \mathbf{2} \text{ years} \\ 1{,}1025 - 1 = 0{,}15763 & \text{after } \mathbf{3} \text{ years} \\ & \ldots \\ 1{,}6289 - 1 = 0{,}62889 & \text{after } \mathbf{10} \text{ years} \end{array}$$

SKETCH OF PROOF.

When interest is compounded yearly, it is added to principal at the end of each year. Therefore at the end of the first year the accrued interest Ci is added to principal which becomes $FV = C + Ci = C(1+i)$. Moreover at the end of second year the accrued interest $C(1+i)i = Ci + Ci^2$, where Ci^2 is **interest on interest**, is added to principal which becomes $FV = C(1+i) + C(1+i)i = C(1+i)^2$. It is readily realised (and proved by **mathematical induction**) that each yearly compounding of interest amounts to a multiplication of the principal by $(1+i)$ so that we get $FV = C(1+i)^t$ at the end of the *t*-th year. Although time *t* is integer in our reasoning, it can take any nonnegative real value owing to the **exponential** convention.

Example 5.

€25.000 are lent for 1 year, 6 months and 18 days at the yearly rate of 6%, as in Example **2**; find accumulation and compound interest under the assumption that each month has 30 days.

Solution.

We have

$$FV = C(1+i)^t = 25.000 * 1{,}06^{1{,}55} = 27.363{,}02 \text{ €}$$

$$I = FV - C = C(1+i)^t - C = 27.363{,}02 - 25.000 = 2.363{,}02 \text{ €}$$

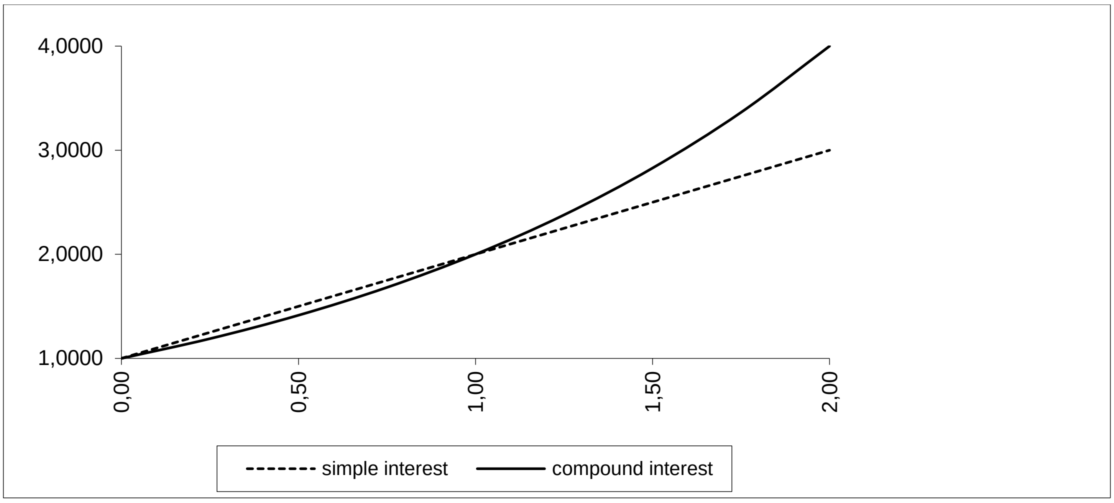
Consider the accumulations under simple and compound interest at the same yearly rate *i*, the accumulation factors being $(1+it)$ and $(1+i)^t$ respectively. As portrayed in the following diagram, where $i = 100\%$, the former displays **linear** growth, whereas the latter displays **exponential** (geometric) growth with

$$(1+it) > (1+i)^t \text{ for all } 0 < t < 1$$

and

$$(1+it)<(1+i)^t \text{ for all } t>1$$

owing to the payment of **interest** on **interest**.



Therefore, for any given yearly interest rate *i* and any term longer than one year, accumulation under compound interest is larger than that under simple interest. For instance, we have $(1+i)^2=1+2i+i^2>1+2i$ for $t=2$, the difference i^2 being the **interest** on **interest**.

When the term *t* is not integer, the **linear** convention (and hence **mixed compounding**) can also be applied, whereby the future value (or accumulation) *FV* at time *t* of a principal C is

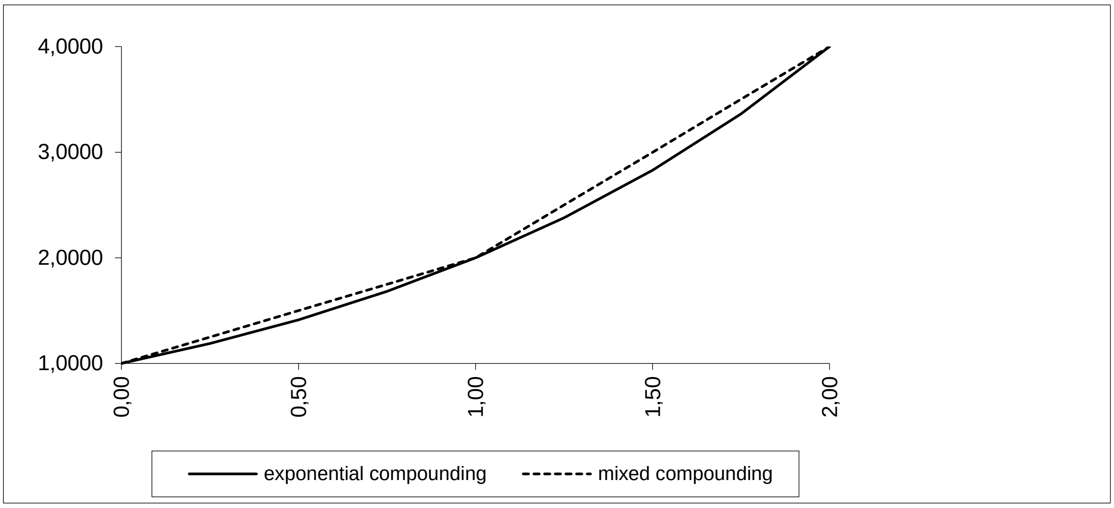
$$FV=C(1+i)^n(1+i\delta)$$

where $t=n+\delta$ with n integer and $0\leq\delta\leq1$. If, for instance, $n=3$ years and $\delta=0,25 \text{ years}=3 \text{ months}$, the accumulation factor is $f(t)=(1+i)^3(1+i0{,}25)$ and stems from the use of compound interest over a 3-year term followed by the use of simple interest over a 3-month term.

Since the exponential function $(1+i)^t$ is **convex**, we have

$$(1+i)^n(1+i\delta)\geq(1+i)^{n+\delta}$$

Therefore, for any integer term ($\delta=0$ and $t=n$) the same accumulation is obtained with either rule; for any non integer term mixed compounding gives a larger accumulation. The graphs of both accumulation factors for $i=100\%$ are plotted in the following diagram



Example 6.

At the beginning of a certain year €5.000 are lent at the yearly rate of 4%, with €2.500 being lent after 9 more months, as in Example **3**. Find accumulation and compound interest after 12 more months by using the **linear convention** and **mixed compounding**.

Solution.

Recall that the future values of the 2 transactions can be summed, as the future value is a **linear** operator in the amounts of money. We have

$$FV = C_A(1+i)\left(1+i\,\frac{9}{12}\right)+C_B\left(1+i\,\frac{3}{12}\right)\left(1+i\,\frac{9}{12}\right)= \ 7.956{,}75\ €$$

$$I = FV-(C_A+C_B)= \ 7.956{,}75-7.500=456{,}75\ €$$

(Zero coupon) bonds

If a loan takes the form of a security, it is divided into a number of bonds so that credit as well as credit risk can be spread across a number of bondholders/lenders. As bonds are securities, credit can be bought and resold. In exchange for credit, the borrower, i.e. the bond issuer, takes the legally binding obligation to make appropriate repayments on stipulated dates. The legal contract between the issuer and each bondholder is called bond indenture. Credit risk is about a financial loss for the bondholders due to the bond issuer defaulting on any contractual repayments.

Bonds are issued by central governments, supranational entities (e.g. World Bank, European Investment Bank, and Asian Development Bank, established respectively in 1944, 1958, and 1966 by a number of member countries, headquartered in Washington, Luxembourg, and Manila), local authorities (e.g. cities), banks, and listed companies among others. As explained in Section 4.4, bonds can also take the form of mortgage-backed securities, which are issued through a securitization process. As explained in Section 4.3, the creditworthiness of bond issuers is assessed by international credit-rating agencies. If the credit rating of a central government is appropriate, Treasury securities can be regarded as being free of credit risk; in contrast, corporate bonds bear credit risk to some extent.

Needless to say, a **bilateral** loan between a single lender, e.g. a commercial bank, and a single borrower is more flexible than a **bond**. If the borrower is a large and important company, a syndicate of lenders, e.g. international banks, may be involved. However, bonds take many different forms, including zero coupon, fixed rate, floating rate ones, introduced below, in Section 4 and 5.3, respectively. Fixed and floating rate bonds pay regular coupons, i.e. interest on the borrowed capital, either yearly, or half-yearly, or quarterly; moreover, they usually repay the borrowed capital in a lump sum at their expiry. Some fixed rate bonds are callable, i.e. they can be redeemed by the issuer before their maturity date at a stipulated call price, possibly including a premium.

A **zero coupon** (or **pure discount**) **bond** pays no coupons so that it is issued and quoted at discount; its market price is thus lower than the face value and equal to its present value, calculated on the basis of a yearly **yield to maturity**. **Settlement lags**, **commissions**, and **taxes** are explicitly considered in Exercise **4**, where the net yield to maturity of some Italian Treasury bills is computed.

Example 7.

An investor subscribes for some zero coupon bonds at issue that have a face value of €10.000, 6 months to maturity, and a price of 98,533 percent. Assume that each month has 30 days and find

a) the invoice price;

b) the yearly **yield to maturity** under the rule of **simple** interest;

c) the yearly **yield to maturity** under the rule of **compound** interest.

Solution.

Let time t be measured in years and y be the unknown yield to maturity.

a) The invoice price is $10.000 * \frac{98{,}533}{100} = 9.853{,}30 \text{ €}$.

b) From the equation

$$PV = \frac{C}{1+yt} \quad \text{i.e.} \quad 98{,}533 = \frac{100}{1+y0{,}5}$$

where $(1+y0{,}5)^{-1}$ is a **half-yearly** discount factor, we get $y = 2\left(\frac{100}{98{,}533} - 1\right) = 2{,}978\%$.

c) From the equation

$$PV = C(1+y)^{-t} \quad \text{i.e.} \quad 98{,}533 = 100(1+y)^{-0{,}5}$$

where $(1+y)^{-0{,}5}$ is a **half-yearly** discount factor, we get $y = \left(\frac{100}{98{,}533}\right)^2 - 1 = 3\%$.

Commercial discount

Let time t be measured in **years**, 0 be the present time, and d be the yearly rate of commercial discount, i.e. the yearly discount on a unit face value. Suppose that a credit C due at time t is sold to a bank at time 0. Since **commercial discount** increases linearly with time by the equation

$$D = Cdt$$

the present value PV

$$PV = C - D = C - Cdt = C(1-dt)$$

is the amount paid by the bank at time 0. Therefore, the present value of C at time 0 is equal to the face value C times the **discount factor** $\frac{1}{f(t)} = (1-dt)$; for PV to be positive we must have $t < \frac{1}{d}$.

Example 8.

A **draft** is a written and legally binding order to a bank to pay the holder a specific face value on a specific and subsequent maturity date. When it is used to pay an invoice in international trade, it is called a **bill of exchange**, which is like a post-dated cheque. If endorsed by a bank, it carries little credit risk and is called a **bank acceptance**.

A manufacturer sells goods on credit now; his foreign customer signs a bill of exchange, the face value being worth €70.000 and due 3 months and 9 days from now. The foreign customer's bank endorses the bill of exchange so that the manufacturer can obtain an early payment of his credit, which is discounted by his bank at the yearly rate of 8%. Find the amount received now by the manufacturer under the assumption that each month has 30 days.

Solution.

Since the time *t* must be measured in years, we have

$$D = Cit = 70.000 * 0{,}08 * \left(\frac{3}{12} + \frac{9}{360} \right) = 70.000 * 0{,}08 * 0{,}275 = 1.540 \text{ €}$$

$$PV = C - D = C(1 - dt) = 70.000 - 1.540 = 70.000(1 - 0{,}08 * 0{,}275) = 68.460 \text{ €}$$

1.2. Financial contracts and markets

The **financial system** of an economy is composed of **financial intermediaries**, **institutional investors**, **financial markets**, and **regulatory authorities**. **Financial intermediaries** include: commercial and investment banks, credit unions, brokers and dealers, consumer credit companies, leasing and factoring companies. **Institutional investors** include: insurance companies, pension funds, investment advisors, mutual funds, real estate investment trusts, hedge funds, private equity and venture capital funds. As for Italy, **regulatory authorities** include: the Bank of Italy, established in 1893, CONSOB (an acronym of Commissione nazionale per le società e la borsa), established in 1974, and IVASS (an acronym of Istituto per la vigilanza sulle assicurazioni), which is supervised by the Bank of Italy.

A **financial system** allows operators to make payments and funds to flow from savers/lenders to spenders/borrowers; more precisely, funds can flow either **indirectly** through **financial intermediaries** and **institutional investors** or **directly** through **financial markets**. Whether financing is direct or indirect, the flow of funds is matched by the stipulation of financial contracts or the trade of securities; in both instances, lenders take some financial risks, such as market and credit risks. Some specific market or credit risks can be mitigated or hedged by entering into appropriate derivative contracts; in other words, they can be transferred to the counterparts of the derivative contracts. Firms are net borrowers, i.e. they have a shortage of funds, whereas households are net savers, i.e. they have excess funds. Governments are borrowers whenever they run budget deficits, with expenditures exceeding tax receipts.

Foreigners can be either net lenders or net borrowers. The above-mentioned processes are supported by sophisticated telematic networks. Indeed, a huge number of payments and a huge number of trades are executed every day in their respective networks.

When financing is direct, firms collect debt and/or equity capital, as their bonds and/or stocks are subscribed at issue. Bonds and stocks are issued on **primary markets** and traded on **secondary markets**. The latter can take the form of an **exchange**, or be **over-the counter**, or take the form of a **multilateral trading facility**, known as **electronic communications network** in the US.

An **exchange**, such as NYSE (an acronym of New York Stock Exchange), LSE (an acronym of London Stock Exchange), and BI (an acronym of Borsa italiana), is a regulated market, designated and overseen by a competent regulatory authority, such as SEC (an acronym of Securities and Exchange Commissions), FSA earlier (an acronym of Financial Services Authority), and CONSOB. Listed stocks are more **liquid** and more **volatile** than unlisted ones. Each listed company meets specific requirements and undergoes a yearly audit. The dealing system of an exchange can be either **order** or **quote** driven; in other words, either buy and sell orders for the listed securities are electronically matched or market makers quote their bid-ask spreads and match orders. A **market order** is executed at the best available price, whereas a **limit order** is executed at the stipulated limit price or better. Order execution is guaranteed only by market makers, who are willing to buy (sell) securities at the bid (ask) price to overcome imbalances between buy and sell orders. A clearing house guarantees the settlement of all trades. Both dealing systems are used by LSE.

An **over-the counter market**, such as NASDAQ earlier (an acronym of National Association of Securities Dealers Automatic Quotations), currency markets, the eurobond market, and the US bond market to a great extent has no physical location and is made up by brokers and dealers connected by phones and computer networks. Phone calls are usually recorded and replayed in case of a dispute over the agreed terms. Each trade occurs directly between a broker and a dealer; however, there is a small risk that the trade is not settled.

A **multilateral trading facility** is a private electronic dealing system on the Internet approved by the competent regulatory authority; the electronic limit-order book is generally displayed to all users.

REMARK.

The transition from large medieval market fairs to Renaissance bourses and exchanges is reconstructed by Poitras (2012), who also explains the origin of the term bourse.

Established in 1817 as the New York Stock and Exchange Board, NYSE is the largest and most important American stock exchange. Established in 1773, LSE is the largest and most important European stock exchange. The global stock market was centred on the latter (former) in the 19th (20th) century, when electric telegraph and telephone were invented. Notably, Paris bourse closed down in 1789 owing to the French Revolution, whereas Amsterdam was occupied by French troops in 1795.

NYSE merged with Euronext in 2007 and acquired AMEX in 2008, whereas LSE merged with BI in the same year. Euronext is a group established in 2000 and made up of the Amsterdam Stock Exchange, Brussels Stock Exchange, Paris Bourse, London International Financial Futures and Options Exchange (since 2002), and Lisbon Stock Exchange (since 2002).

Financial markets include: **money** markets, **capital** markets, derivative markets, currency markets, and commodity markets. Short-term financial contracts with an original term to maturity lower than or equal to 1 year are traded on a **money** market. They include: Treasury bills (see Exercise **4**), repurchase agreements (see Exercise **2**), certificates of deposits (see Exercise **3**), bank acceptances (see Example **8**), commercial papers, as well as interbank deposits and loans, with Libor and Euribor (see Exercise **5**) being the main interbank ask rates. Each **capital** market is divided into 2 sectors: the fixed income market and the stock market. Bonds with an original term to maturity greater than 1 year are traded in the former, whereas stocks are traded in the latter.

Italian Treasury securities are issued through regular electronic tenders conducted by the Bank of Italy and are listed on BI. **Corporate bonds** are either **publicly** or **privately placed**.

A **public offering** may be conducted by a syndicate of merchant/investment/universal banks; the **bookrunner**, or **lead manager**, organises the syndicate of **underwriters** and **sellers** (e.g., in 7-10 days after a firm commitment mandate), keeps the syndicate book of total demand and works together with the issuer to finalise the **prospectus** and determine the **offering price** (e.g., in 5 more business days). The prospectus provides accurate but redundant information about the prospects of the issuer and the terms of the transaction; it must be endorsed by the competent regulatory authority, i.e. CONSOB in Italy. The initial estimate of the offering price is revised, as the syndicate book is updated. The offering price is charged on the primary market (e.g., for 15 more days). Underwriters act on a **firm commitment** or a **best efforts** basis. In the former case, they may buy all newly issued corporate bonds from the issuer at discount and endeavour to resell them at the offering price. Alternatively, underwriters may commit themselves to buying the unsubscribed corporate bonds. At any rate, since the issuer is generally an important company with a good credit rating, they run the underwriting risk, i.e. of a financial loss consequent on an insufficient demand for the corporate bonds at the offering price. In the latter case, underwriters don't run the underwriting risk, as they act as brokers, who simply do their best to sell the entire new issue at the offering price. Underwriters and sellers are paid a **gross spread** by the issuer; e.g. 1% of the public offering, which provides a considerable reward for their direct and indirect marketing efforts. The gross spread is made up of 3 components: **management** fees paid to **lead manager** and **co-lead managers**, **underwriting** fees paid to **underwriters**, and **selling** concessions paid to **sellers**. The same financial intermediary can play more roles. A public placement is typically listed on a stock exchange, with one among the underwriters usually becoming a market maker.

Private placements (e.g. of smaller loans) are simpler, faster, and less expensive, with potential medium- or long-term investors (e.g., banks, insurance companies, pension and mutual funds) being contacted directly by a financial intermediary. Most corporate bonds are privately placed; **eurobonds**, which are worth at least $100 million, are privately placed too by syndicates of international banks. Eurobonds are usually **bearer** bonds.

The main features of **stocks** are outlined in Part II. In an **initial public offering**, a company goes public, i.e. it sells its stocks to the public for the first time. Stocks can be newly issued or not; both of them are usually offered. An initial public offering is typically conducted by a **syndicate** of **underwriters** and **sellers**. If stocks are to be listed on Borsa Italiana, an underwriter must act as a **sponsor**, i.e. a fiduciary of the potential investors, who introduces the company to them and Borsa Italiana. The sponsor must release regular equity research on the company. Both the **lead manager** and **sponsor** may be appointed 4/6 months before the initial public offering. First of all, a timetable of the listing process is settled. Second, the company undergoes due diligence, e.g. for 3 months. Third, potential syndicate members are contacted; for instance, 15-30 financial intermediaries might deal with the general public, whereas 4-6 financial intermediaries might deal with institutional investors. Subsequently, the size of the offering and a preliminary range of offering prices are determined. About 3 months before the initial public offering, an application for listing may be made to Borsa Italiana; several documents are submitted, possibly including the 3 most recent financial statements, a 3-year business plan, and the QMAT, a file on the company's competitive positioning. Moreover, the **prospectus** is submitted to CONSOB, the competent regulatory authority. An **offering circular** for institutional investors may be drawn up as well. The 3 most recent financial statements, the 3-year business plan, and the prospectus must be endorsed by an audit firm. Marketing efforts may start 2 months before the initial public offering. First of all, the company is presented to the security analysts of the syndicate members, who, in turn, carry out, equity research based on **discounted cash flows** and **financial multiples**, which are presented in Section 3.5.

Institutional investors and very wealthy investors are then polled, road shows being planned in accordance with their feedback. Once the prospectus has been endorsed by CONSOB, the syndicate members start building the syndicate book, which records the amounts demanded by each institutional investor at different feasible offering prices. All indications are not binding. Subsequently, road shows are held, usually by the sponsor, e.g. for 2 weeks to publicize the event. The primary market for the general public follows shortly and lasts between 3 and 5 business days. The maximum offering price is made known the day before its start, whereas the actual offering price is set at its end. If the initial public offering is oversubscribed, allotment to the general public is random.

REMARK.

If the initial public offering is oversubscribed, an over-allotment to institutional investors might occur by 10-15% at most. Over-allotted stocks are borrowed for a month. The **lead manager** is given a free **greenshoe** option, i.e. an option to buy the same amount of stocks at the offering price, which expires after a month. There are 2 possible outcomes that depend on the stock price one month after the initial public offering. If the stock price is consistently above the offering price, the lead manager will exercise the greenshoe option. In contrast, if the stock price is below the offering price, the lead manager will attempt to stabilize the market by buying stocks on it. In either case, stocks will be returned to lenders. In the latter case, trade profits will add to the gross spread of the over-allotted stocks.

Underwriters and sellers are paid a **gross spread**, which might range from 1,6% to 8% of the initial public offering. The **lead manager** and **co-lead managers** are paid **management** fees, which might be about 25% of the gross spread. **Sellers** are paid **selling** concessions, which might be about 50% of the gross spread. **Underwriters** are paid **underwriting** fees, which are equal to the remaining 25% of the gross spread.

A **private placement** is the cheaper alternative to an initial public offering. A financial intermediary sells the stocks directly to a small group of institutional or very wealthy investors. Private placements don’t trade on stock exchanges.

Exercise 1.

At the beginning of a certain year €5.000 are paid into an ideal saving account that earns

a) **simple** interest;

b) **compound** interest according to the **exponential** convention;

c) **compound** interest according to the **linear** convention;

at the rate of 4% **per year**. How long does it take for the accumulation to be equal to €7.000?

Solution.

Let time *t* be measured in years.

a) From the equation $7.000 = 5.000(1 + 0,04t)$ we get

$$t = \left(\frac{7.000}{5.000} - 1\right)\frac{1}{0,04} = 10\,\text{years}$$

Therefore, with **simple** interest the unknown term is equal to 10 years.

b) From the equation $7.000 = 5.000(1 + 0,04)^t$ we get

$$t = \frac{\log(7.000/5.000)}{\log(1,04)} = 8,579\,\text{years} = 8\text{ years} + 0,579 * 360\,\text{days} = 8\text{ years and }209\,\text{days}$$

by **rounding up**. Therefore, with **compound** interest in accordance to the **exponential** convention the unknown term is equal to 8 years, 6 months, and 29 days.

c) From the equation $7.000 = 5.000 * 1,04^8(1 + 0,04t) = 6.842,85(1 + 0,04t)$ we get

$$t = \left(\frac{7.000}{6.842,85} - 1\right)\frac{1}{0,04} = 0,574\,\text{years} = 0,574 * 360\,\text{days} = 207\,\text{days}$$

by **rounding up**. Therefore, with **compound** interest in accordance to the **linear** convention the unknown term is equal to 8 years, 6 months, and 27 days.

Exercise 2.

A **repurchase agreement** is a financial agreement that includes a spot sale of securities and their simultaneous forward repurchase; the securities are often bonds that have a suitable credit rating and do not pay any coupons between the 2 settlements. The seller commits to repurchase the same bonds from the buyer on a specific future date, e.g. after 1-6 months, and at a specific dirty price. In other words, the seller borrows money from the buyer against bonds as a collateral (guarantee): if the seller did not meet his obligation, the buyer would withhold the bonds, whereas if the buyer did not meet his obligation, the seller would withhold the amount borrowed.

Repurchase agreements are entered into by companies and financial intermediaries to borrow or lend cash in the short term; moreover, they are also used by Central Banks to influence interest rates.

Consider the following trade: an Italian bank sells some A bonds to a customer for €80.000, with the settlement date following the trade date by 3 business days; the bank commits at the same time to repurchase those bonds 91 days after settlement for €80.875, even if they are in default. Hence, gross interest is worth €875; since it is taxed at source at a 20% rate, net interest and accumulation are €700 and €80.700. Find the **net** rate of interest per year on this transaction; use **simple interest** and the **actual/365** day count convention, as occurs in Italy with repurchase agreements on bonds.

Solution.

Let time be measured in years and *i* be the unknown rate of interest per year. Let $C = 80.000$ and $FV = 80.700$. Since simple interest implies that $\frac{FV}{C} = 1 + i\frac{91}{365}$, we have

$$i = \frac{365}{91}\left(\frac{FV - C}{C}\right) = \frac{365}{91}\frac{I}{C} = 3{,}510\%$$

In other words, the **net** rate of simple interest is 3,510% per year.

REMARK.

The European Central Bank sets 3+1 key interest rates for the €area, on the **deposit** and **marginal lending facilities** as well as on the **main** and **longer-term refinancing operations**, which are **open market operations**. Standing facilities and refinancing operations are used by the ECB to manage liquidity and steer short-term interest rates in accordance with its **monetary policy**; if liquidity is scarce (abundant), short term rates are driven up (down) owing to an imbalance between demand and supply in the **money market**. The main aim of the ECB and its monetary policy is price stability, defined as an inflation rate not greater than, but close to 2% per year, over the **medium term** in the €area. Monetary policy is unlikely to have a direct effect on medium- and long-term interest rates, which affect firms' decisions about investment and households' decisions about homes and durable goods.

The national central banks are willing to take in **overnight** deposits from and provide **overnight** loans to the banking system at the former interest rates. Loans are available against eligible collateral; the banking system includes all institutions that are obliged to hold reserves with their national central banks. The 2 **overnight** interest rates outline a corridor for the **main refinancing** rate; they are normally the worst possible ones in the €zone, serving as a floor and a ceiling for the overnight interbank rate too (as measured by the **EONIA**, an acronym of Euro OverNight Index Average, a **yearly** rate of **simple** interest of the €zone charged according to an actual/360 day count convention; more precisely, each EONIA is a weighted average, calculated by the ECB, of all overnight **unsecured** interbank loans actually provided by a panel of 28 banks).

The latter interest rates apply to regular **open market operations**, initiated by the ECB and executed through standard tenders, which are held on a weekly (monthly) basis for the **repurchase agreements** (or the **secured loans**) with a term of 1 week (3 months). Although bids are submitted to the national central banks, **allotment** decisions are taken by the ECB; in a **fixed** (volume and) **rate** tender all bids are allocated *pro rata*, whereas in a **variable rate** tender only the best bids are allocated at their respective interest rates. Whenever a **main** (or **longer-term**) **refinancing operation** is conducted, the national central banks lend money to the banking system over a 1-week (3-month) term; more precisely, they buy spot and sell forward an appropriate amount of eligible securities, which can be worth more or less than the expiring amount. The **main refinancing operations** normally supply the bulk of liquidity to the banking system; moreover, other open market operations can be carried out by the ECB either on an *ad hoc* basis or more regularly, also including other financial transactions, such as foreign currency swaps and outright purchases and sales of securities.

Nevertheless, when it comes to the daily management of liquidity, banks generally resort to the interbank market, a wholesale OTC market with the amount of any transaction exceeding €1 million. Credit risk is coped with a 2 tier system, whereby larger and more respected banks trade with each other across countries as well as with smaller banks based in their own country.

Exercise 3.

A **certificate of deposit** is a negotiable security tied to a time deposit with a bank. According to a certificate of deposit issued by a bank to an individual investor, a capital of €100.000 turns into a gross accumulation of €140.000 over a 5-year term. Hence, gross interest is worth €40.000; since it is taxed at source at a 12,5% rate, net interest and net accumulation are €35.000 and €135.000. The bank claims that the gross (net) interest rate is 8% (7%). What interest rule is used?

Solution.

Let time be measured in years and i be the unknown rate of interest per year. Let $C = 100.000$ and either $FV = 140.000$ or $FV = 135.000$. If reference is made to the former (latter) future value, a gross (net) rate is obtained.

Compound interest implies that $\frac{FV}{C}=(1+i)^5$ and hence that

$$i=\left(\frac{FV}{C}\right)^{\frac{1}{5}}-1=6{,}961\% \quad (6{,}186\%)$$

Simple interest implies that $\frac{FV}{C}=1+i5$ and hence that

$$i=\frac{1}{5}\left(\frac{FV-C}{C}\right)=\frac{1}{5}\frac{I}{C}=8\% \quad (7\%)$$

Thus, **simple interest** is used, which is favourable to the borrower, namely to the bank in this case.

REMARK.

According to the Italian law decrees No. 66 24/4/2014, No. 138 13/8/2011, No. 323 20/6/1996, interest on certificates of deposit is subject to a **withholding** tax of 26%, 20%, 27%, irrespective of their term.

Exercise 4.

On Monday, January 12th, 2009 the **Buoni Ordinari del Tesoro** 15/1-15/4/2009, i.e. the Italian Treasury bills, are issued through an electronic tender conducted by the Bank of Italy. An individual investor subscribes for those **zero coupon bonds** for a face value of €10.000, namely 10 times the minimum lot size. Their price at issue is 99,587 percent; a withholding tax of 12,5% is imposed at issue, with commissions being 0,10% of the face value. The day count convention is **actual/360**. Find

a) the invoice price;

b) the **gross** and **net** yield to maturity per year under the rule of **compound** interest.

Solution.

Let time *t* be measured in years and *y* be the unknown yield to maturity so that $(1+y)^{-365/360}$ is the yearly discount factor.

a) The subscription price is

$$99{,}587+(100-99{,}587)*0{,}125+100*0{,}0010=99{,}739$$

so that the invoice price is $10.000*\dfrac{99{,}739}{100}=9.973{,}90$ € .

b) As January 15th, 2009 falls on Thursday and April 15th on Wednesday, the **actual** term lasts $16+28+31+15=90$ days. We have

$$99{,}587=100(1+y)^{-90/360}$$

and

$$99{,}739=100(1+y)^{-90/360}$$

and hence the **gross** yield $y=1{,}669\%$ per year as well as the **net** yield $y=1{,}051\%$ per year. For 1,051% to be the **actual** net **yield** per year, the future value of €10.000, due after 3 months, should be reinvested for 9 more months on the same terms.

REMARK.

If the individual investor needed cash before the expiry of his Treasury bills, he could sell them in the secondary market. The settlement date follows the issue or trade date by 2 business days.
BOTs are issued with a term of 3, 6, and 12 months through regular electronic tenders conducted by the Bank of Italy. The tenders for BOTs are **competitive**; in other words, the successful bids for a zero coupon bond are the best made by financial intermediaries, with all successful bids being filled at their respective prices.
Nonetheless, end investors subscribe for BOTs at the **weighted average price** of the tender; with maximum commissions on 3/6/12 month BOTs being 0,05%/0,10%/0,15% of face value.

Exercise 5.

A **Euribor** (Euro interbank offer rate) is a yearly rate of **simple** interest posted in the €zone and applicable to **unsecured** interbank loans with a term ranging from 1, 2, 3 weeks to 1, 2, ... , 12 months; the settlement lag is 2 business days, whereas the day count convention is **actual/360**. The 1 month Euribor posted on Monday, January 12th, 2004 was 2,082% (from il Sole 24 Ore, 13/1/2004). Find

a) the expiry date and the actual term;

b) the monthly accumulation factor and the equivalent rate of compound interest.

Suppose that the same capital was lent again to another bank at the earliest convenience. Find

c) the start date of the second unsecured interbank loan.

Solution.

a) The first loan under examination began on Wednesday, January 14th and ended on Monday, February 16th, since February 14th, 2004 fell on Saturday. The actual term was $17+16=33$ days, as the last (first) day must (not) be taken into account.

b) The monthly accumulation factor is

$$1+0{,}02082*\frac{33}{360}=1{,}00191$$

Let i be the equivalent rate of compound interest on a transaction that went from 14/1/2004 to 16/2/2004; we have

$$1+0{,}02082*\frac{33}{360}=(1+i)^{33/360}$$

from which it follows that $i=\left(1+0{,}02082*\frac{33}{360}\right)^{360/33}-1=2{,}102\%$.

c) The second loan under examination began on Monday, February 16th; the appropriate Euribor charged was posted on Thursday, February 12th.

1.3. Equivalent rates of compound interest

Let time t be measured in years and 0 be the present time. Suppose that a principal C is lent over the term $[0;t]$ and that interest is compounded m times per year at the **periodic** rate $i_m=\frac{j_m}{m}$, with the contractual rate j_m being a **nominal yearly** rate **convertible** m times per year.

Consider the case of a **current account**; although the contractual rate is a yearly rate j_m, interest is compounded at a smaller periodic rate $i_m=\frac{j_m}{m}$: $m=2$ $(m=4)$ implies that interest is compounded **half-yearly** (**quarterly**) at the half-yearly (quarterly) rate $i_2=\frac{j_2}{2}$ $\left(i_4=\frac{j_4}{4}\right)$.

Define i as the rate of compound interest **effective per year**, if it is equivalent to i_m, i.e. if it produces the same interest (and hence the same accumulation) over a one-year term (and hence over any term) without intermediate compoundings. Since under the **exponential** convention, the **future value**, or accumulation, FV of the principal C one year since start is

$$FV = C\left(1+\frac{j_m}{m}\right)^m = C(1+i_m)^m$$

with m compoundings per year, as well as

$$FV = C(1+i)$$

with a single compounding per year, we obtain the following **equivalence relation** between nominal and effective rates of interest

$$\left(1+\frac{j_m}{m}\right)^m = (1+i_m)^m = 1+i$$

with Ci being the interest altogether accrued in the first year.

Example 9.

A saving account earns interest at the nominal rate of 10% per year convertible half-yearly. Find **a**) the **effective** rate of interest per year; **b**) the interest accrued in the first year on a deposit of €1.000. Suppose that interest compounding becomes quarterly with no change in the **effective** rate of interest per year. Find **c**) the new nominal rate of interest.

Solution.

a) We have $j_2 = 10\%$ and $i = \left(1+\frac{j_2}{2}\right)^2 - 1 = 1{,}05^2 - 1 = 10{,}25\%$.

b) The interest accrued in the first year is $1.000i = 1.000 * 0{,}1025 = 102{,}5$ €.

c) We have $1+\frac{j_4}{4} = (1+i)^{\frac{1}{4}}$ and hence $j_4 = 4\left((1+i)^{\frac{1}{4}} - 1\right) = 4\left(1{,}1025^{0{,}25} - 1\right) = 9{,}878\%$.

For any given i it can be ascertained that $j_2 = 2\left[(1+i)^{1/2} - 1\right] < i$ owing to the payment of **interest on interest** and that the sequence $\{j_m\}$ decreases as m increases, its lower limit being

$\delta = \log(1+i)$, i.e. the rate of **continuous compound** interest introduced below. Therefore, all nominal rates are lower than the effective one.

Now consider the ideal case in which the nominal rate is convertible momently, i.e. interest is compounded instantly and continuously. For instance, such an assumption is made when it comes to value some derivative contracts. We have

$$\delta = \lim_{m \to +\infty} j_m = \lim_{m \to +\infty} \frac{(1+i)^{1/m} - 1}{1/m} = \lim_{t \to 0} \frac{(1+i)^t - 1}{t} = \lim_{t \to 0} \frac{(1+i)^t \ln(1+i)}{1} = \ln(1+i) < i$$

with δ being the yearly nominal rate **convertible momently** (or **continuously compounded**) and with

$$e^{\delta t}$$

being the corresponding accumulation factor over the term $[0;t]$.

Example 10.

At the beginning of a certain year €25.000 are paid into an ideal saving account that earns **continuous compound** interest. The accumulation after 2,5 years is €26.917,40. Find

a) the **effective** rate of interest per year;

b) the yearly nominal rate of interest **convertible momently**.

Solution.

Let time t be measured in years. Let $C = 25.000$ €; $FV = 26.917{,}40$ €; $t = 2{,}5$ years. From $FV = C(1+i)^t = Ce^{\delta t}$ it follows that

a) $i = \left(\frac{FV}{C}\right)^{1/t} - 1 = 3\%$;

b) $\delta = \log(1+i) = 2{,}956\%$.

Note that $\delta < i$ as stated above.

It can also be ascertained that the larger i, the larger is the difference $i - \log(1+i)$. Some numerical examples are given in the following table

i	$\delta = \log(1+i)$	$i - \log(1+i)$
1%	0,995%	0,005%
2%	1,980%	0,020%
3%	2,956%	0,044%
4%	3,922%	0,078%
5%	4,879%	0,121%
6%	5,827%	0,173%
7%	6,766%	0,234%
8%	7,696%	0,304%
9%	8,618%	0,382%
10%	9,531%	0,469%

REMARK.

The approximation $\log(1+i) \cong i - \frac{i^2}{2}$ proves fairly accurate for $i \leq 15\%$. Using a Taylor's expansion truncated at 2nd order obtains $\log(1+i) > i - \frac{i^2}{2}$ for $i > 0$.

Principle of consistency

Let time t be measured in suitable units and 0 be the present time. Denote with $f(t)$ an accumulation factor that depends only on the term t rather than on the start and end dates of a financial transaction (e.g., the **start** and **expiry** dates of an **interbank** loan). Consider the following diagram

$$0 \qquad\qquad t \qquad\qquad t+\tau$$

and remember that $f(t+\tau)$ is the future value at time $t+\tau$ of an investment of 1 over the term $[0;t+\tau]$ whereas $f(t)f(\tau)$ is the future value at time $t+\tau$ of an investment of 1 over the term $[0;t]$ followed by a reinvestment of the proceeds over the term $[t;t+\tau]$.

Definition.

The accumulation factor $f(t)$ is **consistent** if

$$f(t)f(\tau) = f(t+\tau) \quad \text{for all } t,\tau \geq 0$$

namely if accumulation is not affected by the course of action taken by the investor.

Let *i* be the **periodic** rate of simple (or compound) interest. We have

$$(1+it)(1+i\tau)=1+i(t+\tau)+(it)(i\tau)\neq 1+i(t+\tau) \quad \text{with simple interest}$$

$$(1+i)^t(1+i)^\tau = (1+i)^{t+\tau} \quad \text{with compound interest}$$

Therefore, the accumulation factor $f(t)=1+it$ based on simple interest is not consistent whereas the accumulation factor $f(t)=(1+i)^t$ based on compound interest is consistent.

Proposition.

A **differentiable** accumulation factor $f(t)$ is consistent **iff** (if and only if) it is such that $f(t)=(1+i)^t$, namely **iff** interest is compounded.

PROOF.

We have $\ln f(t)+\ln f(\tau)=\ln f(t+\tau)$ and hence the Cauchy functional equation $g(t)+g(\tau)=g(t+\tau)$ owing to the substitution $g(t)=\ln f(t)$. For $t=\tau=0$ the Cauchy equation becomes $g(0)+g(0)=g(0)$ and hence $g(0)=0$. Differentiating the Cauchy equation with respect to *t* yields $\frac{dg(t)}{dt}=\frac{dg(t+\tau)}{dt}$ so that $\frac{dg(t)}{dt}=\delta$. Therefore, we have $g(t)=\delta t$, as only a straight line with a null intercept has a constant derivative and is such that $g(0)=0$. Finally, $\ln f(t)=g(t)=\delta t$ is equivalent to $f(t)=e^{\delta t}$, i.e. to an accumulation factor based on **continuous compound** interest at the **nominal** rate δ convertible momently.

When consistency holds, future (and present) values can be calculated in several manners. For instance, since the above definition can be rewritten as

$$f(t)=\frac{f(t+\tau)}{f(\tau)} \quad \text{for all } t,\tau\geq 0$$

the accumulation of 1 over the term $[0;t]$ can also be obtained as the present value at time *t* of the accumulation of 1 over the term $[0;t+\tau]$. This **mathematical** property comes in useful when dealing with annuities, also implying that the comparison of several annuities based on the same rate of interest *i* results in the same ranking irrespective of the appraisal time.

When it comes to an investment in **bonds**, the **yield to maturity** and the **actual yield** do agree with each other only if $f(t)$ is consistent.

Example 11.

An investor buys a **zero coupon bond** with face value of €5.000 and 12 months to maturity. The yield to maturity is 3% per year. The investor resells the bond 8 months later, when the yield to maturity is still 3% per year. Assume **commissions** and **taxes** away; find the actual yield on the transaction under the assumption that the yield to maturity is expressed as a rate of

a) **simple interest**, as is in the case of Italian Treasury bills;
b) **commercial discount**, as is in the case of UK and US Treasury bills;
c) **compound interest**.

Solution.

Let time *t* be measured in years. A zero coupon bond is always quoted at discount; its market price is thus lower than the face value and equal to its present value, calculated on the basis of the yield to maturity.

a) The proper discount factor is $\frac{1}{1+0{,}03t}$, where 3% is the yield to maturity and *t* is the **time to maturity**. Therefore, the buying price is $\frac{5.000}{1+0{,}03*1} = 4.854{,}37$, where 1 year is the remaining time to maturity, whereas the selling price is $\frac{5.000}{1+0{,}03*4/12} = 4.950{,}50$, where 4 months is the remaining time to maturity. The unknown actual yearly yield meets the equation

$$\frac{\text{accumulation}}{\text{principal}} = \frac{\text{selling price}}{\text{buying price}} = \frac{4.950{,}50}{4.854{,}37} = 1 + r\frac{8}{12}$$

from which it follows that $r = 2{,}970\%$, with *r* being a rate of **simple interest**. Since simple interest is not consistent and the investment breaks up before maturity, the actual yield on the transaction is different from the constant yield to maturity.

b) The proper discount factor is $1-0{,}03t$, where 3% is the yield to maturity and *t* is the **time to maturity**. Therefore, the buying price is $5.000(1-0{,}03*1) = 4.850$, whereas the selling price is $5.000\left(1-0{,}03*\frac{4}{12}\right) = 4.950$. The unknown actual yearly yield meets the equation

$$\frac{\text{accumulation}}{\text{principal}} = \frac{\text{selling price}}{\text{buying price}} = \frac{4.950}{4.850} = \frac{1}{1 - d\,\frac{8}{12}}$$

from which it follows that $d = 3{,}030\%$, with d being a rate of commercial discount. Since commercial discount is not consistent and the investment breaks up before maturity, the actual yield on the transaction is different from the constant yield to maturity.

c) The proper discount factor is $1{,}03^{-t}$, where 3% is the yield to maturity and t is the **time to maturity**. Therefore, the buying price is $5.000 * 1{,}03^{-1}$, whereas the selling price is $5.000 * 1{,}03^{-4/12}$. The unknown actual yearly yield meets the equation

$$\frac{\text{accumulation}}{\text{principal}} = \frac{\text{selling price}}{\text{buying price}} = \frac{5.000 * 1{,}03^{-4/12}}{5.000 * 1{,}03^{-1}} = 1{,}03^{8/12} = (1 + r)^{\frac{8}{12}}$$

from which it follows that $r = 3{,}000\%$, with r being a rate of compound interest. Since compound interest is consistent, the actual yield on the transaction is the same as the constant yield to maturity.

No arbitrage with 2-variable accumulation factors

Let time t be measured in suitable units and 0 be the present time. Denote with $f(0;t)$ an accumulation factor that depends on the start and end dates of a financial transaction (e.g., the **start** and **expiry** dates of an **interbank** loan).

Definition.
The **2-variable** accumulation factor $f(t;t+\tau)$ is **consistent** if

$$f(0;t)\, f(t;t+\tau) = f(0;t+\tau) \quad \text{for all } t,\tau \geq 0$$

namely if accumulation is not affected by the course of action taken by the investor.

Proposition.
A **differentiable** accumulation factor $f(t;t+\tau)$ is consistent **iff** (if and only if) it is such that $f(0;t) = \exp\left(\int_0^t \delta(t)\,dt\right)$, namely **iff** interest is **continuously** compounded at the **nominal** rate

$\delta(t)$ convertible momently. The following proof is alternative to the original one by the Italian mathematician Francesco Paolo Cantelli (1875-1966).

PROOF.

Let $T = t + \tau$; we have $\ln f(0;t) + \ln f(t;T) = \ln f(0;T)$ and hence the Cauchy functional equation $g(0;t) + g(t;T) = g(0;T)$ for 2-variable functions owing to the substitution $g(0;t) = \ln f(0;t)$. For $t = T = 0$ the Cauchy equation becomes $g(0;0) + g(0;0) = g(0;0)$ and hence $g(0;0) = 0$. Differentiating the Cauchy equation with respect to T yields $\frac{\partial g(t;T)}{\partial T} = \frac{\partial g(0;T)}{\partial T}$ so that $\frac{\partial g(t;T)}{\partial T} = \delta(T)$, as $\frac{\partial g(0;T)}{\partial T}$ does depend only on *T*. Therefore, we have $g(t;T) = \ln f(t;T) = \int_t^T \delta(T)dT$ and hence $f(0;t) = \exp\left(\int_0^t \delta(t)dt\right)$, i.e. an accumulation factor based on **continuous compound** interest. The consistency of compound interest can also be proved under the more general assumption that all 2-variable accumulation factors are **measurable**.

If the time pattern of $\delta(t)$ is known, we benefit from **certainty**, as all term structures of interest rates are known, both the present and the future ones. More precisely,

- $i_{0;t} = \frac{\exp\left(\int_0^t \delta(t)dt\right)}{t}$ is the current **spot** rate of interest on a financial transaction with term *t*;
- $i_{t;t+\tau} = \frac{\exp\left(\int_t^{t+\tau} \delta(t)dt\right)}{\tau}$ is the **spot** rate of interest charged at time *t* on a financial transaction with term τ;
- if $\delta(t)$ is an increasing / a constant / a decreasing function of time *t*, all the present spot rate of interest $i_{0;t}$ (the future spot rates of interest $i_{t;t+\tau}$) do increase / remain constant / decrease with the term *t* (τ).

Consider an ideal **capital market** where

- there are no **frictions** such as commissions, fees, bid-ask spreads, taxes, margin requirements, restrictions on short sales;
- any amount of money can be either lent or borrowed;

- operators are **rational**, i.e. profit maximisers, and price takers;
- there are no defaults on contractual obligations;
- interest rates, both present and future, are **certain** and **public**.

Remember that an **arbitrage** is a set of **simultaneous** financial transactions that does not require any (down)payment and does or may provide some receipt. It can be ascertained that arbitrage is ruled out in those circumstances, **iff**

- there is a **unique** term structure of interest rates;
- present and future values are **linear** operators in the amounts of money;
- 2-variable accumulation factors are **consistent**.

Exercise 6.

The nominal rate of (compound) interest on a certain bank account is

a) 3,8% per year **convertible quarterly**;

b) 4% per year **convertible quarterly**;

c) 3,8% per year **convertible half-yearly**.

Find the corresponding **effective** rate of interest per year.

Solution.

Consider the formula

$$1+i=\left(1+\frac{j_m}{m}\right)^m$$

where j_m is a nominal yearly rate of interest **convertible *m* times per year** and i is the corresponding **effective** rate of interest per year.

a) Substituting $m=4$ and $j_m = j_4 = 3{,}8\%$ in the previous formula obtains $i = 3{,}854\%$.

b) Substituting $m=4$ and $j_m = j_4 = 4\%$ in the previous formula obtains $i = 4{,}060\%$.

c) Substituting $m=2$ and $j_m = j_2 = 3{,}8\%$ in the previous formula obtains $i = 3{,}836\%$.

Exercise 7.

At the beginning of a certain year €5.000 are paid into a bank account that earns interest at the rate of 4% per year **convertible quarterly**.

a) How long does it take for the accumulation to be equal to €5.500?

b) How much is compound interest?

Solution.

Let time t be measured in years.

a) From the equation $5.500 = 5.000\left(1,01^4\right)^t$ we get

$$t = \log\left(\frac{5.500}{5000}\right)\frac{1}{4\log 1,01} = 2,395 \text{ years} = 2 \text{ years and } 143 \text{days}$$

by **rounding up**. Therefore, the unknown term is equal to 2 years and 143 days.

b) Compound interest is worth $I = FV - C = 5.500 - 5.000 = 500$ €.

Exercise 8.

At the beginning of a certain year €100.000 are paid into a bank account that earns compound interest at the nominal rate of 5% per year **convertible half-yearly**. A 20% **withholding tax** is imposed on interest.

a) Allow for the tax liability and find the equivalent (gross) rate under **yearly compounding**.

b) Suppose that neither the interest nor the tax rate change over time. Find the largest constant amount that can be withdrawn at the end of each half-year over an indefinite span of time.

c) Suppose the above-mentioned withdrawals are made. Find the net accumulation after 3 years and 3 months (hint: use the **linear convention** and hence **mixed compounding**).

Solution.

a) The unknown rate i satisfies the equation

$$\left[1 + \frac{0,05}{2}(1 - 0,2)\right]^2 = 1 + i(1 - 0,2)$$

whereby the net accumulation factor per year is the same in both instances. The solution to this equation is $i = 5,05\%$.

b) The largest possible half-yearly withdrawal is equal to the half-yearly net interest

$$100.000\frac{0,05}{2}(1 - 0,2) = 100.000 * 2\% = 2.000 \text{ €}$$

with 2% being the net interest rate per half-year. Any larger amount would sooner or later empty the bank account.

c) The net accumulation 3 months after each withdrawal, and therefore on the required date, is

$$100.000\left(1+0{,}02\frac{90}{180}\right)=101.000\ €$$

However, the quarterly net interest, worth €1.000, has not been compounded yet.

REMARK.

According to the presidential decree No. 600 29/9/1973 (art. 26, paragraph 2) and subsequent amendments, including the law decree No. 323 20/6/1996 (art. 7, paragraph 6), a 27% withholding tax is imposed at source on interest on savings, current bank and postal accounts by banks and the Italian Post Office. It is a **tax** for individuals, by way of **advance** for entrepreneurs and companies, among others. Such a distinction also applies to the above-mentioned repurchase agreements and certificates of deposit. According to the law decree No. 138 13/8/2011 (art. 2, paragraph 6) the above-mentioned withholding tax is reduced to 20%. According to the law decree No. 66 24/4/2014 (art. 3, paragraph 1) the above-mentioned withholding tax is raised to 26%.

Exercise 9.

A principal of €200.000 is lent from Tuesday, March 1st to Wednesday, June 1st at the **simple** interest rate of 2% per year. The net accumulation is lent again from Wednesday, June 1st to Monday, October 3rd at the **simple** interest rate of 2,2% per year. The day count convention is **actual/365**, whereas interest is taxed at source at a 20% rate. Allow for the tax liability and find the net accumulation and interest of the 2 loans combined.

Solution.

Let time *t* be measured in years and $C=200.000$. Consider the 1st loan. As its **actual** term lasts $30+30+31+1=92$ days, the net accumulation at expiry is

$$FV_1=C\left[1+0{,}02(1-0{,}2)\frac{92}{365}\right]=200.806{,}58€$$

Now consider the 2nd loan. As its **actual** term lasts $29+31+31+30+3=124$ days, the net accumulation at expiry is

$$FV_2 = FV_1\left[1+0{,}022(1-0{,}2)\frac{124}{365}\right] = 202.007{,}24€$$

The net interest of the 2 loans combined is worth $FV_2 - C = 2.007{,}24\ €$.

Exercise 10.

At the beginning of a certain year, a principal of €250.000 is lent for 24 months at the rate of 4% per year **effective**. The accumulation at expiry is lent again for 12 more months at the rate of 3% per year **effective**. Find

a) the accumulation and interest of the 2 loans combined;

b) the rate of return on the 2 loans combined.

Solution.

Let time *t* be measured in years and *r* be the unknown rate of return per year. Let $C = 250.000$.

a) The accumulation after 24 months is

$$FV_1 = C * 1{,}04^2 = 270.400€$$

whereas the accumulation after 36 months is

$$FV_2 = FV_1 * 1{,}03 = 278.512\ €$$

Therefore, the interest of the 2 loans combined is worth $FV_2 - C = 28.512\ €$.

b) The unknown rate of return per year *r* satisfies the following equivalence relation between accumulation factors

$$1{,}04^2 * 1{,}03 = (1+r)^3$$

from which we get the **geometric mean**

$$r = \left[1{,}04^2 * 1{,}03\right]^{\frac{1}{3}} - 1 = 3{,}666\%$$

Whenever interest rates are not constant, the **geometric mean** is lower than the **arithmetic mean** owing to Jensen's inequality

$$r = 3{,}666\% < \frac{2}{3}0{,}04 + \frac{1}{3}0{,}03 = 3{,}667\%$$

Exercise 11.

The rate of interest on a certain bank account is 2,75% per year **effective**. The **withholding tax rate** on interest is 20%. The inflation rate is 1,188% per year. Suppose that neither the interest, nor the tax rate, nor the inflation rate change over time. Find

a) the net real rate of interest per year;

b) the net real accumulation of €10.000 after 7 years.

Solution.

a) Since the net nominal and real accumulations of €1 after 1 year are

$$1+0{,}0275(1-0{,}2)=1{,}022 \quad \text{and} \quad \frac{1{,}022}{1{,}01188}=1{,}01$$

the net real rate of interest is worth $1{,}01-1=1\%$ per year.

b) The net real accumulation after 7 years is $10.000\left(\frac{1{,}022}{1{,}01188}\right)^7 = 10.000*1{,}01^7 = 10.721{,}35$ € .

1.4. Immediate annuities: present values, future values, and values

Let time *t* be measured in years. Suppose that interest is compounded (and payments are discounted) at the rate *i* **per year effective**. An **immediate annuity** agreed upon, or issued, or bought at time 0 is a sequence of *n* **yearly** payments in **arrears**, the first payment $R_1 > 0$ being due after 1 year and the *t*-th payment $R_t > 0$ being due after *t* years, at the end of the *t*-th year. Such sequence of **positive** payments is represented by the following diagram

	R_1	R_2		R_{n-1}	R_n
0	1	2	$\cdots$	$n-1$	n

To enhance comprehension, we make reference to an imaginary but insightful case. A safe bond, issued at time 0, promises to make the above payments. There are no commissions, fees, and taxes. Since the issuer will meet his/her obligations for sure, the financial contract bears no **credit** risk. Nor does it bear any **interest rate** risk, *i* being its yield to maturity at any time until expiry. An investor buys the safe bond at issue and holds it until expiry; each amount R_t is paid into a current account, which earns compound interest at the rate *i* per year.

- The **present value** at time 0 of an immediate annuity is

$$\sum_{k=1}^{n} R_k \left(1+i\right)^{-k} = R_1 \left(1+i\right)^{-1} + R_2 \left(1+i\right)^{-2} + \ldots + R_n \left(1+i\right)^{-n}$$

  namely the sum of the present values of all *n* payments and thus the price at time 0 of the safe financial contract.

- The **future value** or accumulation at time *n* of an immediate annuity is

$$\sum_{k=1}^{n} R_k \left(1+i\right)^{n-k} = R_1 \left(1+i\right)^{n-1} + R_2 \left(1+i\right)^{n-2} + \ldots + R_n$$

  namely the sum of the future values of all *n* payments and thus the balance at time *n* of the current account.

- The **value** at time *t* of an immediate annuity payable **yearly** in **arrears** is

$$\sum_{k \le t} R_k \left(1+i\right)^{t-k} + \sum_{k > t} R_k \left(1+i\right)^{-(k-t)}$$

  namely the future value of all payments due **before** and **at** time *t* plus the present value of all payments due **after** time *t*. Time *t* can take any real value. Therefore, these 2 terms are also the balance at time *t* of the current account and the price at time *t* of the safe financial contract.

Note that

(1) **value** at time 0 $=$ **present value** at time 0

(2) **value** at time n $=$ **future value** at time n

(3) **value** at time $t =$ **value** at time 0 $*\left(1+i\right)^t$

with all equations holding even if the amounts R_t take different signs.

Equation (**3**) is a consequence of **consistency**: first transfer all payments backward in time and compute their (present) value at time 0, then transfer this amount forward in time and obtain their value at time *t*. For $n = 3$ and $t = 2$ we have

$$\left(R_1\left(1+i\right)^{-1} + R_2\left(1+i\right)^{-2} + R_3\left(1+i\right)^{-3}\right)\left(1+i\right)^2 = R_1\left(1+i\right) + R_2 + R_3\left(1+i\right)^{-1}$$

Suppose that **several** (**immediate**) **annuities** have to be compared on the basis of the same rate of interest *i*. This comparison can be carried out at any time owing to equation (**3**) and therefore to **consistency**. If 2 annuities have the same value at a particular time, then they are **financially equivalent** at any time; if an annuity is the largest (smallest) in value at a particular time, then it is so at any time.

REMARK.

The rules of simple interest and commercial discount do not benefit from such an important property; as a consequence, 2 different evaluation times may result in 2 different rankings of the annuities under examination.

Immediate annuities: constant yearly payments

Recall that time t is measured in years and interest is compounded at the rate i **per year effective**. An **immediate level annuity** agreed upon, or issued, or bought at time 0 is a sequence of n **yearly constant** payments in **arrears**, the first payment R being due after 1 year (and the t-th payment R being due after t years). Such sequence of **positive** payments is represented by the following diagram

	R	R		R	R
0	1	2	$\cdots$	$n-1$	n

The **present value** at time 0 of an immediate level annuity is

$$R\left[(1+i)^{-1}+(1+i)^{-2}+\ldots+(1+i)^{-n}\right]=Ra_{n|i}=R\frac{1-(1+i)^{-n}}{i}$$

where the annuity factor $a_{n|i}$ is the present value of a sequence of n yearly unit payments, calculated one year before the first payment is made.

PROOF.

Suppose that a loan of €1 is to be repaid by n yearly instalments in arrears. The first $n-1$ instalments are equal to i, the **interest** for the latest year, whereas the last instalment is equal to $i+1$, the **interest** for the last year plus **capital**. For i to be a yearly rate of compound interest, the loan must be equal to the present value of all instalments: $1=ia_{n|i}+(1+i)^{-n}$, which implies that $a_{n|i}=\frac{1-(1+i)^{-n}}{i}$.

The **future value** or accumulation at time *n* of an immediate level annuity is

$$R\left[(1+i)^{n-1}+(1+i)^{n-2}+\ldots+1\right]=Rs_{n|i}=Ra_{n|i}(1+i)^{n}=R\frac{(1+i)^{n}-1}{i}$$

where the annuity factor $s_{n|i}$ is the future value of a sequence of *n* yearly unit payments, calculated when the last payment is made. The second equality follows from the property (**3**) of immediate annuities.

The **value** at time *t* of an immediate level annuity is the future value of all payments due **before** and **at** time *t* plus the present value of all payments due **after** time *t*

$$R\left(\sum_{k\le t}(1+i)^{t-k}+\sum_{k>t}(1+i)^{-(k-t)}\right)=Ra_{n|i}(1+i)^{t}=R\left(\frac{(1+i)^{t}-(1+i)^{-(n-t)}}{i}\right)=R\left(s_{t|i}+a_{n-t|i}\right)$$

Time *t* can take any real value. However, if time *t* is a **round** number, the future value $Rs_{t|i}$ of *t* payments due **before** and **at** time *t* is added to the present value $Ra_{n-t|i}$ of $n-t$ payments due **after** time *t*. The first equality follows from the property (**3**) of immediate annuities.

Immediate annuities: constant periodic payments

If payments are made $m=2$ (or $m=4$, or $m=12$, or $\ldots$) times per year, work with the half year (or the quarter, or the month, or $\ldots$) as time unit and hence with the period rate i_m such that $(1+i_m)^m=1+i$. For instance,

- $R\dfrac{1-(1+i_2)^{-n}}{i_2}$ is the present value at time 0 of *n* **half-yearly** payments, each worth *R*
- $R\dfrac{(1+i_4)^{n}-1}{i_4}$ is the future value at time *n* of *n* **quarterly** payments, each worth *R*
- $R\left(\dfrac{(1+i_{12})^{t}-(1+i_{12})^{-(n-t)}}{i_{12}}\right)$ is the value at time *t* of *n* **monthly** payments, each worth *R*

Immediate perpetuities

Recall that time *t* is measured in years and interest is compounded at the rate *i* **per year effective**. An **immediate level perpetuity** issued or bought at time 0 is a sequence of an **infinite** number of **yearly constant** payments in **arrears**, the first payment *R* being due after 1 year (and the *t*-th payment *R* being due after *t* years). Such sequence of **positive** payments is represented by the following diagram

	R	R		R	
0	1	2	$\cdots$	t	$\cdots$

The **present value** at time 0 of an immediate level perpetuity is

$$R\left[(1+i)^{-1}+(1+i)^{-2}+\ldots+(1+i)^{-t}+\ldots\right]=R\sum_{t=1}^{\infty}(1+i)^{-t}=Ra_{\infty|i}=R\frac{1}{i}$$

where the annuity factor $a_{\infty|i}$ is the present value of a sequence of an infinite number of yearly unit payments calculated one year before the first payment is made.

PROOF.

We have: $Ra_{\infty|i}=\lim_{n\to+\infty}Ra_{n|i}=\lim_{n\to+\infty}R\frac{1-(1+i)^{-n}}{i}=R\frac{1}{i}$.

An **immediate geometric perpetuity** issued or bought at time 0 is a sequence of an **infinite** number of **yearly** payments in **arrears** growing by a **geometric progression**, the first payment *R* being due after 1 year (and the *t*-th payment $R(1+g)^{t-1}$ being due after *t* years, where *g* is the yearly rate of growth). Such sequence of **positive** payments is represented by the following diagram

	R	$R(1+g)$	$R(1+g)^2$		$R(1+g)^{t-1}$	
0	1	2	3	$\ldots$	t	$\ldots$

The **present value** at time 0 of an immediate geometric perpetuity is

$$R\left[\frac{1}{1+i}+\frac{1+g}{(1+i)^2}+\frac{(1+g)^2}{(1+i)^3}+\ldots+\frac{(1+g)^{t-1}}{(1+i)^t}+\ldots\right]=R\sum_{t=1}^{\infty}\frac{(1+g)^{t-1}}{(1+i)^t}=R\frac{1}{i-g}$$

where $i>g$ for our geometric series to converge. The following proof is new and alternative to the original one by the Swiss mathematician Leonhard Euler (1707-1783).

PROOF.
We have

$$PV_1 = R\left[\frac{1+g}{1+i} + \left(\frac{1+g}{1+i}\right)^2 + \left(\frac{1+g}{1+i}\right)^3 + \ldots + \left(\frac{1+g}{1+i}\right)^{t-1} + \ldots\right] =$$

$$= R(1+g)\left[\frac{1}{1+i} + \frac{1+g}{(1+i)^2} + \frac{(1+g)^2}{(1+i)^3} + \ldots + \frac{(1+g)^{t-2}}{(1+i)^{t-1}} + \ldots\right] = (1+g)PV_0$$

where PV_0 (PV_1) denotes the present value at time 0 (1) of the subsequent payments. If the above process is further applied, we readily obtain

$$PV_t = (1+g)^t PV_0$$

Moreover, the property (**3**) of immediate annuities implies that

$$R + PV_1 = (1+i)PV_0$$

It follows that: $R + (1+g)PV_0 = (1+i)PV_0$ and hence, by simplifying: $PV_t = \frac{(1+g)^t R}{i-g}$, where $i > g$ for the outcome to be positive and for our geometric series to converge. Indeed, if the previous inequality weren't met, the term $R\frac{(1+g)^{t-1}}{(1+i)^t}$ wouldn't be infinitesimal, as *t* tends to infinity.

Exercise 12.
An **immediate** level annuity includes 5 yearly payments of €100 each. On the basis of an interest rate of 4% **per year effective**, find

a) the present value of the annuity at issue;
b) the future value of the annuity immediately after the last payment;
c) the value of the annuity 3 years after issue;
d) the value of the annuity 3 years and 3 months after issue.

Solution.
Let time be measured in years. The sequence of payments is represented by the following diagram

amount		100	100	100	100	100
time	0	1	2	3	4	5

a) The present value of the annuity at issue is

$$PV_0 = 100 a_{5|4\%} = 100 \frac{1 - 1{,}04^{-5}}{0{,}04} = 445{,}18 \text{ €}$$

Keep in mind that $a_{5|4\%}$ is the present value at a 4% rate of 5 periodic unit payments, calculated one period before the first payment is made.

b) The last payment will be made at time 5. The future value of the annuity 5 years after issue is

$$FV_5 = 100 s_{5|4\%} = 100 \frac{1{,}04^5 - 1}{0{,}04} = 541{,}63 \text{ €}$$

namely the accumulation of 5 payments. Keep in mind that $s_{5|4\%}$ is the future value at a 4% rate of 5 periodic unit payments, calculated at the time the last payment is made.

c) The value of the annuity 3 years after issue is

$$V_3 = FV_3 + PV_3 = 100 s_{3|4\%} + 100 a_{2|4\%} = 500{,}77 \text{ €}$$

namely the accumulation of 3 payments plus the present value of 2 subsequent payments.

d) The value of the annuity 3 years and 3 months after issue is

$$V_{3,25} = 1{,}04^{0,25} V_3 = 1{,}04^{0,25} \left(100 s_{3|4\%} + 100 a_{2|4\%} \right) = 505{,}70 \text{ €}$$

We have $V_{3,25} = 1{,}04^{3,25} PV_0 = 100 s_{3,25|4\%} + 100 a_{1,75|4\%}$ as well; however, the 2 annuity factors $s_{3,25|4\%}$ and $a_{1,75|4\%}$ are not susceptible of a financial interpretation.

REMARK.

Recall that $s_{5|4\%} = 1{,}04^5 a_{5|4\%}$ and $s_{3|4\%} + a_{2|4\%} = 1{,}04^3 a_{5|4\%}$ owing to consistency, which proves helpful when checking for computation errors.

Exercise 13.

A level annuity includes 6 yearly payments of €90 each, the first payment being due 3 years from now. On the basis of an interest rate of 5% **per year effective**, find

a) the present value of the annuity now;

b) the future value of the annuity 1 year after the last payment;

c) the value of the annuity 6 years from now;

d) the value of the annuity 6 years and 6 months from now.

Solution.

Let time be measured in years. The sequence of payments is represented by the following diagram

amount				90	90	90	90	90	90	
time	0	1	2	3	4	5	6	7	8	9

a) The present value of the **deferred** annuity is now

$$PV_0 = PV_2 * 1{,}05^{-2} = \left(90 a_{6|5\%}\right) 1{,}05^{-2} = 414{,}34\ €$$

Keep in mind that $a_{6|5\%}$ is the present value at a 5% rate of 6 periodic unit payments, calculated one period before the first payment is made.

b) The last payment will be made at time 8. The future value of the annuity 9 years from now is

$$FV_9 = FV_8 * 1{,}05 = \left(90 s_{6|5\%}\right) 1{,}05 = 642{,}78\ €$$

Keep in mind that $s_{6|5\%}$ is the future value at a 5% rate of 6 periodic unit payments, calculated at the time the last payment is made.

c) The value of the annuity 6 years from now is

$$V_6 = FV_6 + PV_6 = 90 s_{4|5\%} + 90 a_{2|5\%} = 555{,}26\ €$$

namely the accumulation of 4 payments plus the present value of 2 subsequent payments.

d) The value of the annuity 6 years and 6 months from now is

$$V_{6,5} = 1{,}05^{0,5} V_6 = 1{,}05^{0,5}\left(90 s_{4|5\%} + 90 a_{2|5\%}\right) = 568{,}97 \ €$$

Exercise 14.

An immediate level annuity of €100 **per month** is payable in arrears for 3 years. Find the present value of the annuity 1 month before the first payment is made on the basis of an interest rate of 12% **per year**

a) effective;

b) convertible monthly.

Solution.

Let time be measured in **months**. Since the annuity includes 36 monthly payments in arrears of €100 each, the unknown present value satisfies the equation

$$100 \ a_{36|i_{12}} = 100 \frac{1-\left(1+i_{12}\right)^{-36}}{i_{12}}$$

where i_{12} is the interest rate **per month**.

a) We have $i_{12} = 1{,}12^{1/12} - 1 = 0{,}949\%$ and hence $100 a_{36|0,949\%} = 3.037{,}41 \ €$

b) We have $i_{12} = \frac{12\%}{12} = 1\%$ and hence $100 a_{36|1\%} = 3.010{,}75 \ €$.

Exercise 15.

Suppose that

- at the end of each half year an individual pays €30.000 into a bank account that earns interest at the nominal rate of 4,5% per year **convertible half-yearly**. The **withholding tax rate** on interest is 20%;
- neither the interest nor the tax rate changes over time.

How long does it take for the accumulation to be worth €400.000? (hint: use **mixed compounding**)

Solution.

The net interest rate **per half year** is $i_2 = \frac{0,045}{2}(1-0,2) = 1,8\%$. The number of required payments n meets the equation $30.000 s_{n|1,8\%} = 400.000$. It follows that $1,018^n = 1 + \frac{40}{3} 0,018$ and hence that

$$n = \frac{\log\left(1 + \frac{40}{3} 0,018\right)}{\log(1,018)} = 12,058$$

Therefore, the individual should make 12 half-yearly payments, which result in an accumulation of $30.000 s_{12|1,8\%} = 397.867,55€$ 6 years since start, i.e. 5,5 years since the first payment. As a consequence, 6 years and t days are needed to achieve the target, with t meeting the equation $397.867,55\left(1 + 0,018 \frac{t}{180}\right) = 400.000$ owing to mixed compounding.

The solution to that equation is

$$t = \left(\frac{400.000}{397.867,55} - 1\right) \frac{180}{0,018} = 54 \text{ days}$$

2. Progressive repayment of a loan

2.1. Repayment schedule

Consider the case in which a principal (or capital) C is lent at time 0 and repaid in n **regular** instalments over the term $[0; n]$ as shown below

amount	$-C$	R_1	R_2		R_{n-1}	R_n
time	0	1	2	$\cdots$	$n-1$	n

This may be the case of a **home** loan taken out in **Italy**. It is usually a **mortgage loan** with a term often ranging from 7 to 20 years; the amount lent could not be greater than 75% of the home value, as appraised by an expert. The home is used as a **collateral guarantee**: if the borrower defaults on the repayments, the lender will foreclose on the mortgage and sell the home. However, the borrower must have repayment ability, his net income being in line with the repayments he is going to make. For instance, if a teacher is to pay constant monthly instalments, their amount may not be greater than 33% of the teacher's net monthly income. If the constraint were not met, a relative or friend could stand a **surety** for the teacher, namely a **personal guarantee** backed up by their own possessions.

Each instalment R_t is made up of two components: the interest I_t for the latest period and the capital repayment C_t, i.e. $R_t = I_t + C_t$ for $t = 1,2,\ldots,n$. The distinction between interest and capital is relevant for taxation purposes (the interest paid by the borrower might be tax deductible and the interest received by the lender is usually taxable income) as well as in case of default by the borrower.

In order to take out a 2-5 year personal loan, a borrower can also sign a **promissory note** and an **endorser** can provide an additional **personal guarantee**.

REMARK.

Before approving the teacher's mortgage loan application, the bank has to check his repayment ability and to appraise the market value of his property. The teacher's repayment ability as well as the risks of the banking transaction depend mostly on his net income, expenses, wealth, financial gearing, character and commitment. The most recent personal **tax return** reports his present income and property, whereas expenses may be conventionally set at 67% of his net income. Since 1964 the **Central credit register** collects confidential data on the debts of the customers of all banks and financial companies supervised by the Bank of Italy. All adherents report every month to the Bank of Italy the name of each customer and their overall debt, whenever it is greater than or equal to €30.000; they make also known all present defaults and all defaults they have just incurred. 40 days after the end of each month or so, all adherents can search the archive for the overall debt of an entity. The credits provided to (guarantees given in favour of) each censed customer are grouped into 5 (2) categories; a distinction is drawn between promised and used credit, with a negative difference being an overdraft. The lists of formal protests about promissory notes, accepted and non-accepted bills of exchange, and bank cheques are recorded every month in a **Computerised register of protests** by the Chambers of Commerce. The record on any single protest is stored for 5 years; nonetheless, it can be erased if a promissory note or bill of exchange is paid within 12 months since dishonour is attested.

Possibly by resorting to a third party, one can search the Computerised register of protests for an entity.

Let time be measured in years and i be the rate of compound interest per year effective. The case of a **fixed** rate and **yearly** instalments is considered in the sequel; however, if i is replaced with the periodic rate i_m, the same mathematical process applies to the case of m **periodic** payments per year. The aim is to draw up the **repayment** (or amortisation) **schedule**, a table that includes 5 columns reporting the time patterns of the instalment R_t, the interest I_t, the capital repayment C_t, and the outstanding loan OL_t, i.e. the residual debt at time t. To do so use can be made of the following 3 equations

$$I_t = iOL_{t-1}$$

which states that interest I_t due for the t-th period is a function of the outstanding loan at the beginning of the period,

$$C_t = R_t - I_t$$

an accounting identity, and

$$OL_t = OL_{t-1} - C_t \quad \text{with} \quad OL_0 = C$$

which states that each capital repayment C_t brings down the outstanding loan OL_{t-1}. The repayment schedule can be filled progressively, using these equations repeatedly, starting from row 1 and moving from one row to the immediately lower one. Therefore, for $t=1$ we have: $I_1 = iOL_0 = iC$, $C_1 = R_1 - I_1 = R_1 - iC$, and $OL_1 = OL_0 - C_1 = C - (R_1 - iC) = (1+i)C - R_1$. Now the same process can be repeated for $t=2$ and then for $t=3$, etc., the outcome being the following table

time t, end of the t-th year	instalment R_t	interest I_t	capital repaid C_t	outstanding loan OL_t
0				C
1	R_1	$I_1 = iC$	$C_1 = R_1 - iC$	$OL_1 = (1+i)C - R_1$
2	R_2	…	…	…
…	…	…	…	…
n	R_n	…	…	0

For $OL_n = 0$, i.e. for the loan to be entirely repaid at time *n*, the sequence of instalments $\{R_1, R_2, \ldots, R_n\}$ must be financially equivalent to the amount lent $OL_0 = C$. There are different manners to impose this constraint, which are all equivalent owing to the assumption of compound interest. The simplest one is called **initial closure condition** and requires that the present value of all instalments, calculated at time 0 and at the rate *i*, is equal to the amount lent

$$\sum_{t=1}^{n} R_t (1+i)^{-t} = C$$

As proved in Exercise **17** point **c**, if the initial closure condition is met, the outstanding loan OL_t is the present value of all the remaining instalments at any time *t*, with $0 \le t \le n$. Moreover, it is readily ascertained that $OL_t = OL_0 - (C_1 + C_2 + \ldots + C_t)$. The former (latter) is the most important (simplest) **prospective** (**retrospective**) espression of the outstanding loan OL_t.

Exercise **18** and Exercise **20** deal with **French-type amortisation** and **floating rate amortisation**, both very frequently used in practice. **French-type amortisation** requires a fixed rate *i* and constant instalments *R* so that the initial closure condition simplifies as

$$R a_{n|i} = C$$

Moreover, as proved in Exercise **18** point **c**, capital repayments increase exponentially with time by the equation $C_t = (1+i)^{t-1} C_1$. **Floating rate amortization** allows for a floating rate of interest. In its simplest yet commonest form, the time patterns of capital repayment and outstanding loan are set once and for all at negotiation time $(t = 0)$; more precisely, all *n* instalments are supposed to be constant and such that their present value at the **initial** interest rate is equal to the amount lent *C*. In contrast, the remainder of the repayment schedule is drawn up with time; more precisely, one additional row of the repayment schedule is filled immediately after a payment, as the interest rate is reset and the interest and instalment for the subsequent period are calculated.

Exercise **19** deals with **Italian-type amortisation**, less frequently used in practice. It requires a fixed rate *i* and constant capital repayments $\overline{C}$ so that the **elementary closure condition** simplifies as

$$\sum_{t=1}^{n} C_t = n\overline{C} = C$$

Moreover, as proved in Exercise **19** point **c**, both interest and instalment decrease linearly with time, the former by the equation $I_t = i\overline{C}(n-t+1)$.

REMARK.

Consider a home loan taken out by an Italian family. French-type amortisation is applied with *m* periodic and **constant** instalments *R* per year being due for $\frac{n}{m}$ years; for instance, constant monthly instalments *R* may be due for 15 years $(m = 12 \text{ e } n = 180)$. We have $Ra_{\overline{180}|i12} = C$, where the contractual rate of interest j_{12}, a nominal rate convertible $m = 12$ times per year, may be equal to the $\frac{n}{m} = 15$ year swap rate, quoted at negotiation time, plus a 3% spread.

If, *coeteris paribus*, **floating** instalments are considered, the **nominal** rate **convertible monthly** used to determine the interest I_{t+1}, due at the end of the $t+1$-th month (e.g. the thirteenth month), may be equal to the 1 month Euribor, quoted the last working day of the previous month (e.g. the twelfth month), plus a 3% spread.

Swap and Euribor rates are introduced in Section 5.1.

REMARK.

In the sequel, a realistic example is considered of how commissions, fees, and taxes affect the interest rate charged by the lender.

2.2. Leasing

A **lease** is a contractual arrangement whereby a **lessor** rents some real assets of his to a **lessee** over the term $[0;n]$ in return for a sequence of $n+1$ **regular** rentals $\{R_0; R_1; R_2; \cdots; R_n\}$, with R_0 being paid in advance. At the end of the lease term, the lessee can

- return the real assets to the lessor, e.g. because they don't suit his needs any longer;
- buy the real assets by paying the redemption price $\overline{R}$, a fraction of their value PV_0 at the start of the lease contract;
- renew the lease contract.

Let i_m be the periodic rate of compound interest charged by the lessor, we have

$$PV_0 = \sum_{k=0}^{n} R_k (1+i_m)^{-k} + \overline{R}(1+i_m)^{-n}$$

and hence

$$PV_0 = R_0 + Ra_{n|i_m} + \overline{R}(1+i_m)^{-n}$$

whenever rentals are constant.

Operating leasing was the first to take place, the real assets being very expensive high tech equipment, possibly prone to fast obsolescence, such as large electronic computers. The lessor was a manufacturing company that took the real asset back at the end of the lease.

Financial leasing developed later, the lessor being a finance company that acts as an intermediary between manufacturers or suppliers and asset users. The lessee can exercise a purchase option at the end of the lease. Financial leasing contracts are flexible and can be tailored to individual needs. In a **direct lease** a lessor buys a real asset from, say, a manufacturing company and then rents it to a lessee. In a **sale and lease-back lease** a lessee sells, say, a building of his to a lessor and then rents it from the lessor.

REMARK.

A company may lease a portion of its car and lorry fleet and let the specialist lessor carry out the maintenance. In order to cope with a fluctuating demand, most airlines lease a proportion of their aircraft fleets on a **cancellable basis**; specialist lessors are able to lease again any aircraft returned to them. As a consequence, aircraft owned by specialist lessors are likely to fly more than aircraft owned by airlines.

As for finance companies operating in Italy, an initial outlay PV_0 is followed by a sequence of regular receipts $\{R_0; R_1; R_2; \cdots; R_n\}$ and $\overline{R}$; first the real asset is bought, then rentals and, possibly, the redemption price $\overline{R}$ are received. The real asset is depreciated; each rental is a **revenue**, whereas a positive (negative) difference between the redemption price and the net book value of the real asset is a **capital gain** (**loss**). As for lessees, the rentals $\{R_0; R_1; R_2; \cdots; R_n\}$ may be tax deductible **costs**, provided that the lease term is consistent with the asset life, whereas the redemption price is an **investment** that can be depreciated.

The 2 equations above enable

- a finance company to determine all rentals as a function of i_m, the periodic rate of interest, provided that the real assets can be sold at the redemption price $\overline{R}$, if they are returned by the lessee. The manufacturer/supplier of the real assets may be required to buy them back by a contractual provision. The redemption price $\overline{R}$ may be a small percentage of PV_0;

- a potential lessee to determine the periodic rate of interest i_m as a function of all rentals. Said rate can't be compared with that of a bank loan, if the potential lessee is also willing to borrow money and purchase the real assets. Indeed, a different taxation applies to the rentals of a lease and the instalments of a bank loan, which allows the owner of the real assets to benefit from their depreciation. A method of comparing the 2 alternatives is reported in Benninga (2000, chapt. 5).

Example 12.
A new industrial building worth €1.500.000 is rented by a lessor to a lessee for 15 years in return for a down payment of €300.000 as well as a sequence of constant quarterly rentals in arrears. The nominal interest rate charged by the lessor is 5% per year **convertible quarterly**. The lessee may buy the real asset at the expiry of the lease contract for a redemption price of €150.000. Find **a**) the constant quarterly rental; **b**) the outstanding loan after 5 years.

Solution.
Let time be measured in quarters and 0 be the present time. The equivalent rate **per quarter** is

$$i_4 = \frac{5\%}{4} = 1{,}25\%.$$

a) We have

$$PV_0 = R_0 + Ra_{n|i_4} + \overline{R}(1+i_4)^{-n}$$

where $n+1 = 61$ is the number of rentals, $PV_0 = 1.500.000$ € is the asset value, $R_0 = 0{,}2PV_0 = 300.000$ € is the down payment, R is the unknown quarterly rental, and $\overline{R} = 0{,}1PV_0 = 150.000$ € is the redemption price. Solving the above equation for R obtains

$$R = \frac{PV_0 - R_0 - \overline{R}(1+i_4)^{-n}}{a_{n|i_4}} = 26.854{,}43 \text{ €}$$

b) Let OL_{20} be the outstanding loan after 20 quarters. We have

$$OL_{20} = Ra_{40|i_4} + \overline{R}(1+i_4)^{-40} = 932.528,93 \text{ €}$$

Keep in mind that when it comes to a lease contract, each rental cannot be split into interest and capital repaid.

Exercise 16.

A loan of €500.000 is to be repaid for 7 years by 84 **monthly** instalments in **arrears**. An **interest-only** term of 2 years brings the payment down in the first 2 years so that the borrower has time to raise their income. A larger constant instalment is then paid over an **interest-and-principal** term of 5 more years. On the basis of a nominal yearly rate of 6% **convertible monthly**, find

a) the interest-only payment as well as the interest-and-principal payment;

b) the loan outstanding immediately after the 48th payment.

Suppose that after the 48th payment the term of the loan is extended by 2 years so as to reduce the monthly payment. On the basis of the original interest rate, find

c) the amount of the revised monthly payment.

Solution.

a) The equivalent rate **per month** is $i_{12} = \frac{6\%}{12} = 0{,}5\%$ so that the **interest-only** payment is

$$500.000 * 0{,}005 = 2.500{,}00 \text{ €}$$

whereas the **interest-and-principal** payment is

$$\frac{500.000}{a_{60|0{,}5\%}} = 9.666{,}40 \text{ €}$$

b) The loan outstanding immediately after the 48th monthly payment is

$$9.666{,}40 a_{36|0{,}5\%} = 317.744{,}39 \text{ €}$$

namely the present value then of all remaining 36 monthly payments.

c) For the outstanding loan to be unchanged, the revised monthly payment must be

$$\frac{317.744{,}39}{a_{60|0{,}5\%}} = 6.142{,}89 \text{ €}$$

Exercise 17.

An individual needs a peculiar bank loan. He would like to repay it over 3 years and supposes that he could pay €14.580 after 1 and 2 years, €25.194,24 after 3 years. The rate of compound interest is 8% **per year effective**.

a) How large a principal can he borrow?

b) Construct the repayment schedule.

c) Consider a repayment schedule as that under examination and prove that the initial closure condition, i.e. outstanding loan equal to the present value of all (remaining) payments, is met at any time.

Solution.

Let time t be measured in years and 0 be the present time.

a) Owing to the **initial closure condition** the unknown principal C is equal to the present value at time 0 of all payments

$$C = \frac{14.580}{1{,}08} + \frac{14.580}{1{,}08^2} + \frac{25.194{,}24}{1{,}08^3} = 46.000\ €$$

b) The repayment schedule is reported in the following table

time t, end of the t-th year	instalment R_t	interest $I_t = 0{,}08 OL_{t-1}$	capital repaid $C_t = R_t - I_t$	outstanding loan $OL_t = OL_{t-1} - C_t$
0				46.000
1	14.580	3.680	10.900	35.100
2	14.580	2.808	11.772	23.328
3	25.194,24	1.866,24	23.328	0

To compile the table, let $t = 1$: from $OL_0 = 46.000$ € calculate $I_1 = 0{,}08 OL_0 = 3.680$ € and then $C_1 = R_1 - I_1 = 10.900$ € eventually obtaining $OL_1 = OL_0 - C_1 = 35.100$ €. Now repeat the process, first at $t = 2$ and then at $t = 3$.

c) Let C be still the amount borrowed and i be the rate of compound interest per year effective in use. The entire sequence of instalments is reported in the following diagram

$-C$	R_1	R_2		R_t		R_n
0	1	2	$\cdots$	t	$\cdots$	n

Owing to consistency the payment R_1 at time 1 is, for instance, equivalent to the payment of its future value $(1+i)^{t-1} R_1$ at time *t*; as a consequence the outstanding loan OL_t at time *t* is also equal to the difference between the future value at time *t* of the initial outstanding loan OL_0 and the future value at time *t* of all instalments due between time 1 and time *t*. It follows that

$$OL_t = (1+i)^t OL_0 - \left[(1+i)^{t-1} R_1 + (1+i)^{t-2} R_2 + \ldots + R_t\right] =$$

$$= (1+i)^t \left[(1+i)^{-1} R_1 + (1+i)^{-2} R_2 + \ldots + (1+i)^{-n} R_n\right] - \left[(1+i)^{t-1} R_1 + (1+i)^{t-2} R_2 + \ldots + R_t\right] =$$

$$= \left[(1+i)^{t-1} R_1 + \ldots + R_t + (1+i)^{-1} R_{t+1} + \ldots + (1+i)^{-(n-t)} R_n\right] - \left[(1+i)^{t-1} R_1 + (1+i)^{t-2} R_2 + \ldots + R_t\right] =$$

$$= \left[(1+i)^{-1} R_{t+1} + (1+i)^{-2} R_{t+2} + \ldots + (1+i)^{-(n-t)} R_n\right]$$

and hence that the outstanding loan OL_t at time *t* is also equal to the present value at time *t* of all instalments due between time $t+1$ and time *n*.

Exercise 18.

A loan of €16.000 is repayable for 1 year by constant quarterly instalments in arrears calculated on the basis of a nominal yearly rate of 8% **convertible quarterly**.

a) Find the effective rate per year and construct the **French-type** repayment schedule.

b) Consider the remaining instalments immediately after the second payment. Extract their present value from the repayment schedule.

c) Prove that capital repayments increase exponentially with time.

d) Show how capital repayment and interest can be computed at any time *t* without drawing up a repayment schedule.

Solution.

Let time *t* be measured in quarters.

a) The equivalent rate **per quarter** is $i_4 = \frac{8\%}{4} = 2\%$ whereas the effective rate per year is $i = 1{,}02^4 - 1 = 1{,}08243 - 1 = 8{,}243\%$.

Owing to the **initial closure condition** the amount lent is the present value at time 0 of all instalments, i.e. $16.000 = Ra_{4|2\%}$, so that the quarterly instalment is

$$R = \frac{16.000}{a_{4|2\%}} = 4.201{,}98\,€$$

The repayment schedule is reported in the following table

time t, end of the t-th quarter	instalment R	interest $I_t = i_4 OL_{t-1}$	capital repaid $C_t = R - I_t$	outstanding loan $OL_t = OL_{t-1} - C_t$
0				16.000,00
1	4.201,98	320,00	3.881,98	12.118,02
2	4.201,98	242,36	3.959,62	8.158,40
3	4.201,98	163,17	4.038,81	4.119,59
4	4.201,98	82,39	4.119,59	0,00

To compile the table, let $t = 1$: from $OL_0 = 16.000$ € calculate $I_1 = 0{,}02 OL_0 = 320$ € and then $C_1 = R - I_1 = 3.881{,}98$ € eventually obtaining $OL_1 = OL_0 - C_1 = 12.118{,}02$ €. Now repeat the process, first at $t = 2$, then at $t = 3$ and eventually at $t = 4$.

b) Owing to consistency the outstanding loan OL_t at time *t* is also equal to the difference between the future value at time *t* of the initial outstanding loan OL_0 and the future value at time *t* of all instalments due between time 1 and time *t*. It follows that

$$OL_t = (1+i_4)^t OL_0 - Rs_{t|i_4} = (1+i_4)^t Ra_{n|i_4} - Rs_{t|i_4} = R\left(s_{t|i_4} + a_{n-t|i_4}\right) - Rs_{t|i_4} = Ra_{n-t|i_4}$$

and hence that the initial closure condition, i.e. outstanding loan OL_t equal to the present value at time *t* of all remaining payments, is met at any time *t*. Therefore, the present value sought is $OL_2 = 8.158{,}40\,€$.

c) Recall that *R* is the constant instalment, I_t and C_t are the interest due and the capital repaid at time *t*, OL_t is the outstanding loan at time *t*. From $C_{t+1} + I_{t+1} = R = C_t + I_t$ we get $C_{t+1} + i_4 OL_t = C_{t+1} + i_4(OL_{t-1} - C_t) = C_t + i_4 OL_{t-1}$ and hence $C_{t+1} = (1+i_4)C_t$. Since the solution of the latest equation is $C_t = (1+i_4)^{t-1} C_1$, capital repayments increase exponentially with time. This property comes in useful when checking a repayment schedule for errors; for instance, in the above table we have $C_4 = (1+i_4)^3 C_1$, i.e. $4.119{,}59 = 1{,}02^3 * 3.881{,}98$, as required by theory.

d) As capital repayments increase exponentially with time, we have $C_t = (1+i_4)^{t-1} C_1$ as well as $I_t = R - C_t$, where $C_1 = R - I_1$ and $I_1 = i_4 OL_0 = i_4 C$.

Exercise 19.

A loan of €16.000 is repayable for 1 year by constant quarterly capital repayments in arrears calculated on the basis of a nominal yearly rate of 8% **convertible quarterly**.

a) Construct the **Italian-type** repayment schedule.

b) Consider the remaining instalments immediately after the third payment. Extract their present value from the repayment schedule.

c) Prove that interest decreases linearly with time.

Solution.

Let time *t* be measured in quarters.

a) Owing to the **elementary closure condition** the amount lent is the sum of all capital repayments, i.e. $16.000 = C_1 + C_2 + C_3 + C_4 = 4\overline{C}$, so that the quarterly instalment is $\overline{C} = 4.000$ €. The repayment schedule is reported in the following table

time *t*, end of the *t*-th quarter	capital repaid $\overline{C}$	interest $I_t = i_4 OL_{t-1}$	instalment $R_t = \overline{C} + I_t$	outstanding loan $OL_t = OL_{t-1} - \overline{C}$
0				16.000
1	4.000	320	4.320	12.000
2	4.000	240	4.240	8.000
3	4.000	160	4.160	4.000
4	4.000	80	4.080	0

To compile the table, let $t = 1$: from $OL_0 = 16.000$ € calculate $I_1 = 0{,}02 OL_0 = 320$ € and then $R_1 = \overline{C} + I_1 = 4.320$ € as well as $OL_1 = OL_0 - \overline{C} = 12.000$ €. Now repeat the process, first at $t = 2$, then at $t = 3$ and eventually at $t = 4$.

b) From $OL_t = OL_{t-1} - C_t$ we get $OL_t = OL_{t-1} - (R_t - I_t) = (1+i_4) OL_{t-1} - R_t$ and hence the equation $OL_{t-1} = \dfrac{OL_t + R_t}{1+i_4}$. Moving backward in time obtains the sequence

$$\left\{ OL_n = 0; OL_{n-1} = \frac{R_n}{1+i_4}; OL_{n-2} = \frac{R_{n-1}}{1+i_4} + \frac{R_n}{(1+i_4)^2}; \cdots; OL_0 = C = \sum_{t=1}^{n} R_t (1+i_4)^{-t} \right\}$$

Therefore, the **elementary closure condition** implies the **initial closure condition**; since the latter holds at any time, the present value sought is $OL_3 = 4.000$ € .

c) Recall that $\overline{C}$ is the constant capital repayment, I_t and R_t are the interest due and the instalment paid at time *t*, OL_t is the outstanding loan at time *t*. From $OL_t = OL_{t-1} - \overline{C}$ we get $OL_t = OL_0 - \overline{C}t = C - \overline{C}t$ and hence $OL_t = \overline{C}(n-t)$ as well as $OL_t = C\left(\frac{n-t}{n}\right)$.

Therefore, we have $I_t = i_4 OL_{t-1} = i_4\overline{C}(n-t+1)$ so that interest decreases linearly with time. This property comes in useful when checking a repayment schedule for errors; for instance, in the above table we have $I_3 = i_4\overline{C} * 2$, i.e. $160 = 0{,}02 * 4.000 * 2$, as required by theory.

REMARK.

The loans of the 2 previous exercises differ only in the amortisation procedure, of a French type in Exercise **18** and an Italian type in Exercise **19**. If reference is made to this case and the notion of a convex function, it is readily proved that Italian-type amortisation has

- a greater initial instalment R_1 as well as a lower final instalment R_n;
- a lower total interest. As shown by the 2 repayment schedules, for $t = 1$ Italian-type interest is the same as French-type interest, whereas for $1 < t \le n$ each Italian-type interest is lower than the corresponding French-type interest (hint: we have $C\left(\frac{n-t}{n}\right) < Ra_{n-t|i_4}$ for $1 \le t < n$).

Exercise 20.

A floating rate loan of €34.000 is repayable for 2 years by **half-yearly** instalments in arrears.

a) Apply French-type amortisation and find the first half-yearly payment on the basis of a nominal yearly rate of 6% **convertible half-yearly**.

b) Find the time patterns of outstanding loan and capital repaid.

Suppose that the nominal yearly rate of interest is reset only once to 6,50% **convertible half-yearly**, immediately after the second payment.

c) Complete the repayment schedule.

Solution.

Let time *t* be measured in half-years.

a) The first half-yearly payment is worth

$$R_1 = \frac{34.000}{a_{4|3\%}} = 9.146,92\,€$$

b) Let $t = 1$: from $OL_0 = 34.000$ € we calculate $I_1 = 0,03 OL_0 = 1.020$ € and then $C_1 = R - I_1 = 8.126,92$ € eventually obtaining $OL_1 = OL_0 - C_1 = 25.873,08$ €. As capital repayments increase exponentially with time, we have $C_{t+1} = 1,03 C_t$ so that

time t, end of the t-th period	instalment R_t	interest I_t	capital repaid C_t	outstanding loan $OL_t = OL_{t-1} - C_t$
0				34.000,00
1	9.146,92	1.020,00	8.126,92	25.873,08
2			8.370,73	17.502,35
3			8.621,85	8.880,50
4			8.880,50	0,00

c) The repayment schedule is reported in the following table, where ${}_2i_{0;1} = {}_2i_{1;2} = 3\%$ and ${}_2i_{2;3} = {}_2i_{3;4} = 3,25\%$.

time t, end of the t-th period	instalment $R_t = C_t + I_t$	interest $I_t = {}_2i_{t-1;t} OL_{t-1}$	capital repaid C_t	outstanding loan $OL_t = OL_{t-1} - C_t$
0				34.000,00
1	9.146,92	1.020,00	8.126,92	25.873,08
2	9.146,92	776,19	8.370,73	17.502,35
3	9.190,68	568,83	8.621,85	8.880,50
4	9.169,12	288,62	8.880,50	0,00

Notice that $3,25\% > 3\%$ entails that $R_3 > R_1 = R_2$; in other words, an increase (a decrease) in the half-yearly rate of interest is matched by an increase (a decrease) in the half-yearly instalment, as intuition would suggest.

REMARK.

The outstanding loan OL_t is unfortunately other than the present value at time t and at the periodic rate ${}_mi_{t;t+1}$ of all $n-t$ remaining instalments $\{R_{t+1}; R_{t+2}; \cdots; R_n\}$, unless a theoretically sounder approach is followed, whereby the time patterns of capital repayment and outstanding loan are not set once and for all at negotiation time. The instalment R_{t+1} as well as its breakdown into interest I_{t+1} and capital repaid C_{t+1} are calculated at negotiation time $(t=0)$ as well as at any reset time, immediately after a payment. In doing so, all remaining instalments are supposed to be constant and determined so that their present value at the new interest rate $R_{t+1}a_{n-t|m}\, i_{t;t+1}$ is equal to the outstanding loan $OL_t = C-(C_1+C_2+\cdots+C_t)$, reported in the previous row of the repayment schedule.

Exercise 21.

A loan of €200.000 is to be repaid by 10 constant yearly instalments in arrears that are calculated on the basis of an interest rate of 4% per year effective. Immediately after the 7th payment the borrower considers whether to settle the loan. He has 2 alternatives

a) settling the loan by paying 101% of the outstanding loan;

b) investing the same amount at the rate of 6% per year effective.

Find the best alternative.

Solution.

Let time *t* be measured in years.

a) The yearly constant instalment is worth

$$R = \frac{200.000}{a_{10|4\%}} = 24.658{,}19\,€$$

Since the loan outstanding immediately after the 7th payment is the present value of the remaining instalments

$$OL_7 = Ra_{3|4\%} = 68.428{,}72\ €$$

the early repayment is worth $1{,}01 OL_7 = 69.113{,}01\ €$.

b) If immediately after the 7th payment an amount of $F_7 = Ra_{3|6\%} = 65.911{,}64\ €$ were invested in a fund at the rate of 6% per year effective, all remaining instalments could be paid by drawing from the fund *F*. As *F* obeys the equation

$$F_{t+1} = 1{,}06F_t - R$$

we have $F_t < OL_t < 1{,}01OL_t$ for $t = 7{,}8{,}9$ as well as $F_{10} = OL_{10} = 0$. Nonetheless, the actual investment will be larger and equal to €69.113,01, with the excess

$1{,}01OL_7 - F_7 = 3.201{,}37\,€$ turning into an accumulation of $3.201{,}37 * 1{,}06^3 = 3.812{,}88\,€$ immediately after the 10th payment.

Therefore, the early repayment of the loan is the worst alternative since it does not provide any accumulation.

Exercise 22.

A car worth €50.000 is to be leased for *n* months. The lessee is willing to make a down payment of €5.000 followed by a sequence of monthly rentals in arrears, each worth €775 at most. Moreover, he is likely to buy the car at the expiry of the lease contract for a redemption price of €500. The nominal interest rate charged by the lessor is 4,80% per year **convertible monthly**. Find the unknown term of the lease.

Solution.

Let time be measured in months and 0 be the present time. The equivalent rate **per month** is $i_{12} = \frac{4{,}80\%}{12} = 0{,}40\%.$ We have

$$PV_0 = R_0 + R\frac{1-(1+i_{12})^{-n}}{i_{12}} + \overline{R}(1+i_{12})^{-n}$$

where $n+1$ is the unknown number of rentals, $PV_0 = 50.000\ €$ is the asset value, $R_0 = 0{,}1PV_0 = 5.000\ €$ is the down payment, $R = 775\ €$ is the maximum monthly rental, and $\overline{R} = 0{,}01PV_0 = 500\ €$ is the redemption price. Solving the above equation for *n* obtains

$$(1+i_{12})^n = \frac{\frac{R}{i_{12}} - \overline{R}}{\frac{R}{i_{12}} + R_0 - PV_0} \quad \text{and hence} \quad n = \ln\left(\frac{\frac{R}{i_{12}} - \overline{R}}{\frac{R}{i_{12}} + R_0 - PV_0}\right)\frac{1}{\ln(1+i_{12})} = 65{,}56$$

Therefore, 66 monthly rentals, each worth

$$R = \frac{PV_0 - R_0 - \overline{R} * 1{,}004^{-66}}{a_{66|0{,}4\%}} = 770{,}49\ €$$

must be paid over a term of 5 years and 6 months.

2.3. Italian legislation on consumer credit and mortgage loans

As explained in McCutcheon and Scott (1986, p. 255), "in recent years the governments of various countries have enacted laws aimed at making people

- who borrow money;
- or buy goods or services on credit;

more aware of the true cost of credit and enabling them to compare the true interest rates implicit in various lending schemes. Examples of laws of this type are the *Consumer Credit Act* 1974 in the UK and the *Consumer Credit Protection Act* 1968 in the US. … Regulations made under powers in the *Consumer Credit Act* 1974 lay down what items should be treated as entering into the total charge for credit and how the rate of charge for credit should be calculated. The rate is known as the *Annual Percentual Rate of Charge* (APR) and is defined in such a way as to be the effective annual rate of interest on the transaction, obtained by solving the appropriate equation of value, taking into account all the items entering the total charge for credit. The total charge for credit and the APR have to be disclosed in **advertisements** and in **quotations** for consumer credit agreements."

As for Italy, when it comes to

- **sales on credit** (of cars, furniture, domestic appliances, …) to a **consumer**, namely a natural person who doesn't borrow money as a businessman, a practitioner or an entrepreneur;
- a personal loan taken out by a consumer;
- a **mortgage loan** taken out by either a **natural** or a **legal person**;

among others, 2 financial indicators must be stated, named **TAN** (yearly nominal rate) and **TAEG** (yearly effective global rate) in the first 2 instances, **TAN** and **ISC** (synthetic cost indicator) in the third one. All of this stems from the following legislation

- the law No. 142 19/2/1992 including, among others, some articles on **consumer credit**, passed in compliance with 2 directives by the EEC (now EU);
- the law No. 154 17/2/1992 on trasparency in the field of financial services;
- the decree issued by the minister of Treasury on 8/7/1992 and the decree issued by the governor of Bank of Italy on 24/5/1992 in compliance with the law No. 154 17/2/1992;
- the legislative decree No. 385 1/9/1993 on **banks and credit** as well as the legislative decree No. 58 24/2/1998 on **financial intermediation**, with the former repealing the relevant articles of the first 2 laws of the list;
- the decree issued by the governor of Bank of Italy on 25/7/2003 in compliance with the decree issued by CICR on 4/3/2003;

- subsequent amendments and other additions.

The law No. 108 7/3/1996 on **fight against usury** is among the additional legislation and sets a ceiling to the rate of interest on a loan. More precisely, at the beginning of each quarter the ministry of Treasury records the average (yearly) effective global rate, which includes all charges but taxes and has been applied by banks and financial intermediaries to loans of the same kind during the previous quarter. The various average (yearly) effective global rates are published in the Gazzetta Ufficiale by the end of the collection quarter; when increased by one half, they become the limit beyond which the crime of usury is committed during the subsequent quarter. If usurious interest is agreed, the provision is invalid and interest must be paid at the legal rate.

REMARK.

The term of a consumer credit transaction may range from 2 to 5 years, with the amount lent being not greater than €31.000; the term of a mortgage loan might vary between 7 and 20 years. In the former case, no **collateral** guarantee is required; however, a consumer credit transaction can rest on either a promissory note signed by the borrower or a surety stood for the borrower. It can also be secured by an **assignment** of one fifth of salary or pension, paid by the consumer's employer or pension provider. In that circumstance, the risk of injury, death or job loss has to be appropriately hedged.

Definition.

The **yearly nominal rate** is the yearly **internal** rate of interest for a transaction, calculated on the loan gross amount. It determines, as a function of the loan amount and term, interest and capital repaid for each instalment included in the repayment schedule. It takes both **interest** and **credit risk** into account.

Definition.

The **yearly effective global rate / synthetic cost indicator** is the yearly **internal** rate of interest for a transaction, when account is taken of the additional charges incurred by the borrower, such as a commitment fee, an appraisal fee, and a compulsory insurance premium. The commitment (appraisal) fee matches the expense incurred by the bank or financial intermediary when managing the loan request (estimating the market value of a property). Any tax must be disregarded. The **TAEG/ISC** should be rounded to 2 decimal places, with any intermediate outcome including 8 decimal places; the day count convention is **actual/365**.

REMARK.

An **Italian** contract of mortgage loan must be drawn up by a **notary**, who performs its registration (and its discharge on request) with the **Conservatoria dei registri immobiliari**, one of the 2 sections of the Italian Land Registry. The **legal duration** of a mortgage is 20 years. There can be several **mortgage liens** on a property, with the oldest lien holder being paid off first in a foreclosure proceeding. The ISC doesn't take notary's expenses into account.

Example 13.

A **home loan** of €250.000 was repaid over 20 years by 240 monthly constant instalments in **arrears**, each worth €1.541,43. When the loan was disbursed, the borrower paid the bank: a commitment fee of €875 (0,35% of the amount borrowed), an appraisal fee of €500, a tax of €625 (0,25% of the amount borrowed). Choose the month as unit of time and find TAN and ISC for the mortgage loan under examination.

Solution.

To find the TAN *i*, first calculate the monthly rate of compound interest i_{12} that meets the equation

$$250.000 = 1.541,43 a_{240|i_{12}}$$

whereby when disbursed, the **gross loan** is the same as the present value of all contractual instalments. The **yearly nominal rate** (**convertible monthly**) is equal to $i = i_{12} * 12$. Using the built-in iterative procedure of a spreadsheet package yields $i_{12} = 0,35\%$ and $i = 4,20\%$.

To find the ISC *i*, first calculate the monthly rate of compound interest i_{12} that meets the equation

$$250.000 - 875 - 500 = 248.625 = 1.541,43 a_{240|i_{12}}$$

whereby when disbursed, the **loan before taxes** is the same as the present value of all contractual instalments (a charge of €2 for the instalment collection should be added to each instalment). The **synthetic cost indicator** is equal to $i = (1 + i_{12})^{12} - 1$, provided that i_{12} includes 8 decimal places. Using the built-in iterative procedure of a spreadsheet package yields $i_{12} = 0,355\%$ and $i = 4,35\%$.

The **net loan** amounts to $250.000 - 875 - 500 - 625 = 248.000$ €.

REMARK.

As for the bank, one expenditure is followed by several receipts. Therefore, as explained in Section 3.3, the **internal** rate of interest numerically determined in both instances is unique.

In Exercise **23** the TAEG of a personal loan is computed.

Exercise 23.

A **personal loan** of €11.000 was made by a bank to a pensioner and was secured by the assignment of one fifth of the pension. The personal loan was repaid over 5 years by 60 monthly constant instalments **in arrears**, calculated on the basis of a TAN of 5,40%. When the personal loan was disbursed, the pensioner paid the bank a commitment fee of €120 and a compulsory life insurance premium of €880. The pensioner also paid the fee for a preliminary medical examination. Find the TAEG stated in the contract entered into by the pensioner.

Solution.

Let time *t* be measured in years, *i* denote the unknown TAEG, and i_{12} denote the equivalent rate per month. The monthly instalment is worth

$$R = \frac{11.000}{a_{60|0,45\%}} = 209{,}61 \text{ €}$$

From the equation

$$11.000 - 120 - 880 = 10.000 = 209{,}61 a_{60|i_{12}}$$

first we get $i_{12} = 0{,}785\%$ by using the built-in iterative procedure of a spreadsheet package. Proceeding with the required accuracy obtains $i = (1 + i_{12})^{12} - 1 = 9{,}83\%$.

The **net loan** amounts to $11.000 - 120 - 880 = 10.000$ €.

3. Real investment appraisal

3.1. Use of *pro-forma* financial statements

There can be different reasons why a manufacturing company undertakes a specific **real investment project**; for instance, a factory can be built in a foreign location either to expand sales, because the foreign country represents an interesting market, or to bring production costs down, because the foreign country provides a cheap but skilled labour force and/or favourable taxation, or to acquire know how, because the factory district is very technologically advanced. More generally, the **managerial**, **technical**, **commercial**, and **financial** performance of a company follows from the implementation of a **competitive strategy**, which, in turn, must be consistent with the company organisation, the **industry structure** as well as the relevant **s**ocial, **t**echnical, **e**conomic/**e**cological, and **p**olitical trends.

At any rate, when it comes to appraising a **real investment project** made by a company, many different aspects must be taken into account in a thorough analysis, the examination of which is beyond the scope of this section. It suffices here to say that consistency is required between past and future, the company **managerial**, **technical**, and **commercial** know how as well as its **financial** track record on the one hand, the **competitive strategy** and the attendant plans on the other one. Needless to say, a financial assessment is part of the analysis and often based on a sequence of *pro-forma* financial statements, with marketing and operational/personnel plans being a source of data, as for revenues and costs respectively. If a business plan is drawn up, a section of it has to be devoted to the **simulation** of financial statements. A very concise outline of the underlying method is given below; the interested reader might consult Benninga-Sarig (1997) for a thorough presentation.

Each financial statement includes two main constructs, a simplified income statement and a simplified balance sheet, which are simulated for n business years in a row, e.g. $n=3$-5 and even $n=10$, when it comes to drawing up a corporate **strategic** plan. In doing so the management of excess cash, a **financial activity**, is not taken into account, since attention is devoted only to **business activities**. For each future business year under examination a simplified cash flow statement is then projected, which displays either a **free cash flow** or an **equity cash flow** as the final item. In principle, if use is made of the former (latter), the point of view of lenders and shareholders (shareholders only) is taken.

REMARK.

If the real investment project is to be undertaken by an existing company, all entries of each simplified income statement, balance sheet, and cash flow statement are **incremental**.

Financial indicators such as the **net present value,** the **internal rate of return**, the **profitability index**, and the **payback period** are calculated relative to the projected sequence of **yearly** free or equity cash flows. Let x_t be the projected cash flow for the *t*-th year; if 0 is the appraisal time, and *n* is the time horizon, the relevant sequence is

	x_1	x_2		x_{n-1}	x_n
0	1	2	$\cdots$	$n-1$	n

where the last amount x_n is the sum of a cash flow and a **terminal value**. As a general rule, *pro-forma* financial statements are projected as long as each of them captures some features peculiar to the corresponding business year, whereas the terminal value may be provided by a **dividend discount model** under the conventional assumption that for $t > \tilde{n} \geq n$ the real investment project is in a **steady state** with free (equity) cash flows growing at a constant long-run rate, in line with the average growth rate per year of the underlying economy. If the real investment project is financially sound, all the above-mentioned financial indicators will point that out.

When projecting a *pro-forma* income statement and balance sheet, some entries play the role of pivot elements, as other entries are expressed as a percentage of theirs. For instance, inventory and receivables may be expressed as a percentage of the proceeds of sales, whereas payables may be expressed as a percentage of total costs; all percentages may be equal to appropriate historical averages.

Keep in mind that the income statement is compiled on an **accrual basis**, whereas the cash flow statement is compiled on a **cash basis**. The inputs to the cash flow statement of the *t*-th year are some items of the income statement of the *t*-th year as well as the changes in some items of the balance sheet of the *t*-th year with respect to the previous business year. The procedure for obtaining either an **equity cash flow** or a **free cash flow** is sketched in the following table, where, e.g., $\Delta\text{equity}(t) = \text{equity}(t) - \text{equity}(t-1)$, equity being raised for $\Delta\text{equity}(t) > 0$ and stocks being bought back for $\Delta\text{equity}(t) < 0$. As for the accounting adjustments, recall that **depreciation** is added back to net income, since it is a cost not matched by an expenditure, i.e. a noncash item; an increase in **inventories** is taken away from net income, since it is an expenditure not matched by a cost, whereas a decrease in **inventories** is added to net income, since it is a cost not matched by an expenditure; an increase in **receivables** or **accrued revenues** is taken away from net income, since it concerns revenues not matched by receipts yet; an increase in **payables**, **accrued costs** or **taxes payables** is added to net income, since it concerns costs not matched by expenditures yet; an increase in **prepaid expenses** (e.g., due to insurance premia) is taken away from net income, since it is an expenditure not matched by a cost, whereas an increase in **unearned revenues** (e.g., due to fees received in advance) is added to net income, since it is a receipt not matched by a revenue.

+	net income
+	depreciation
–	Δnet working capital
=	**cash from operations**
–	investment
+	Δdebt
=	**equity cash flow** (= dividend – Δequity + Δcash)
+	after tax interest
–	Δdebt
=	**free cash flow**

Reselling fixed assets (property, plant, equipment) is assumed away; if it weren't, account would be taken of capital gains and losses.

REMARK.

Suppose that an international power and gas company is considering whether to build a new green power station. Since its track record is a very helpful source of data, it may set $n = 10$.

REMARK.

A **business plan** is a planning tool focused on the medium term, i.e. a time period of 3-5 years. It may be drawn up, whenever a business is to be successfully run, equity or debt capital is to be raised, a new and challenging project is to be undertaken, be it a real investment, or a merge and acquisition, or a restructuring one. A business plan must be sober and concise as well as well structured, relevant and exhaustive. Each statement must be supported by precise data and detailed information; sources must be mentioned. The different copies may be numbered so as to facilitate any request of return.

Business administration may rely on a **rolling** business plan, updated once a year and complemented with the budget for the subsequent business year. Its drafting is an iterative process with possible implications for the corporate competitive strategy; it may start in early summer and end in fall. When planning, goals and guidelines are **prospectively** set and agreed upon, with corporate resources being accordingly allocated to business units; a viable implementation follows from an in-depth and shared analysis. When checking a business plan, an actual performance is compared with a projected one, thus shedding light on organisational strengths and weaknesses; moreover, forecast and management skills are **retrospectively** assessed over a period of time.

Suppose that a new challenging real investment project is to be undertaken by a manufacturing start up. As explained by Ford et al. (2007), the **business plan** is (30-50 pages long and) likely to be divided into the following sections:

1) table of contents;

2) executive summary: company profile and position (mission, total number of employees, location, products/markets/technologies, **key figures**, key owners/managers), competitive strategy (vision, milestones, differentiating features, capital requirements, **key figures**);

3) concise **qualitative** description of the company: mission, vision, goals and objectives, history;

4) products and services: main features, use and appeal, stage of development, intellectual property;

5) marketing plan: market analysis (main trends, segmentation, actual and potential customers), industry (main trends, concentration, product differentiation, barriers to entry) and competitive analysis (actual and potential competitors, their possible moves), swot analysis (with a breakdown by function), marketing strategy (4 P's: price, product, promotion, place), including sales forecasts;

6) operational plan: product development, properties and facilities, suppliers, business processes and costs, inventory management, quality management, customer service, maintenance, relevant regulations;

7) organisation and management: key owners/managers and their resumes, key advisors, organisational structure, personnel plan;

8) financial structure: legal form of the start up, ownership and financial structure, capital requirements;

9) financial plan: track record (3 years at least), key assumptions on the future performance, *pro forma* financial statements, financial indicators and accounting ratios.

The executive summary is the crucial section, which must be written last. Indeed, experts often discard a business plan without reading beyond the executive summary.

REMARK.

When examining an industry structure, an educated use can be made of the notions of

- **life cycle** of a **product/industry**, made up of the 4 stages of initial development, expansion and consolidation, maturity, decline, with early failures occurring in a large proportion in the first stage, liquidity possibly lacking in the second stage, and high efficiency being essential in the third stage, where process innovation is important. Mature companies have often proved unable to adapt to a radical innovation. **Web services** have shown a different life cycle. As for **assembled products**, the number of manufacturers may grow over time in the first stage, as experimentation takes place, and peaks around the appearance of a **dominant design**, with survival odds being possibly more favourable for more experienced entrants. The industry may then be exposed to a shakeout and become much more concentrated and more homogeneous as a consequence, with its leaders beginning to emerge and survival odds turning much more unfavourable for new entrants. As shown by Suárez-Utterback (1995), the above proposition was met in the US by the life cycles of typewriters, cars, televisions, picture tubes, and transistors, with their dominant designs coming up in 1906, 1923, 1952, 1956, 1959.
- **Porter's 5 competitive forces**, i.e. **1**) suppliers and their bargaining powers, **2**) distribution channels and customers, **3**) industry concentration and competitors, **4**) barriers to entry and potential competitors, **5**) substitute products. In a more **concentrated industry** companies have larger sizes and benefit from higher profitabilities, easier funding (blue chips, especially), more opportunities of learning by doing, with wider R&D activities being possible. **Barriers to entry** follow from such differentiating features as brand (i.e. product quality and range, customer service), scale and scope economies (due to combined mass production and mass distribution), diversification, R&D, managerial, technical, and commercial know how as well as industrial secrets and patents. Technical and managerial know how has paved the way for the worldwide successes of German and Japanese manufacturing as well as of US producers after World War II, the former mostly in more traditional industrial sectors, the latter in more innovative ones too, e.g. chips, software, biopharmaceuticals.

3.2. Net present value (NPV)

Let time *t* be measured in years and time 0 be the appraisal date; let a **real investment project** be represented by a stream of forecast (**yearly**) **expenditures** x_t^-, where $x_t^- < 0$, and (**yearly**) **receipts** x_t^+, where $x_t^+ > 0$.

x_0	x_1	x_2		x_{n-1}	x_n
0	1	2	$\cdots$	$n-1$	n

The **net present value** at time 0 of our investment project is

$$PV_0 = \sum_{t=0}^{n} x_t (1+r)^{-t}$$

where *r* is the **required rate** of (compound) **return**, per year as default. Our setting is **semideterministic**, as all cash flows are **semicertain**. Although they are represented as if they were certain, they are not so. As explained in Cuthbertson-Nitzsche (2001, p. 82), a **sensitivity analysis** can be performed to have a feel for the impact of uncertainty on the net present value PV_0, which, for instance, can be calculated 3 times with reference to either a pessimistic, or an intermediate, or an optimistic scenario. A similar approach can be taken when assessing the internal rate of return, the profitability index, or the payback period of the investment project.

It is readily ascertained that the net present value is a **linear** operator: if the investment project *A* with net present value $PV_{0;A}$ and the investment project *B* with present value $PV_{0;B}$ are carried out together, the resulting net present value is

$$PV_{0;A+B} = PV_{0;A} + PV_{0;B}$$

Moreover, if we double all the cash flows of the investment project *A*, the resulting net present value is

$$PV_{0;2A} = 2PV_{0;A}$$

As shown in Exercise **24**, the required rate of return *r* has a twofold meaning, being both a **cost of capital** and a **reinvestment rate**. Each company must reward its stakeholders at a yearly rate called **cost of capital** that depends on the **business** under examination, the company **financial record** and the company **financial structure**. If x_t is a **free** (**equity**) cash flow, *r* is a cost of

total capital (of **equity capital**), i.e. the rate of return required by stockholders and lenders (stockholders). Said rate can be assessed through a cross-section analysis of some (**listed**) companies that are **comparable** in terms of business (industry), technology and customers, as outlined in Part II and more thoroughly examined by Benninga-Sarig (1997, chapt. 9). An overview is provided by Table 1, where some **empirical** estimates of the cost of equity **after corporate taxes** are reported; reference is made to an ideally **ungeared** firm, with **industry** and **firm size** being the 2 determinants.

REMARK.

If the real investment project is to be undertaken by a start up and x_t is a **free** (**equity**) cash flow, the net present value PV_0 is equal to the **market value** of its **total capital** (of its **equity capital**), also known as **enterprise value**. If the real investment project is to be undertaken by an existing company, such market values are **incremental**.

Safe rate of interest, per year	4%
Low risk industries (e.g. utilities, banks, insurance companies)	6-7%
Medium risk industries (i.e. mature ones, with an average sensitivity to the business cycle)	8-9%
High risk industries (i.e. technically advanced ones)	10-12%
Small firms in a **mature** industry	13-15%
Small firms (start ups too) in an **innovative** industry	15-20%

Table 1 – Empirical estimates of the yearly **cost of equity** for an ideally **ungeared** firm
(from: Massari, M, Zanetti, L 2004 *Valutazione finanziaria,* McGraw Hill, Milano, chapt. 5)

As remarked by Luenberger (1998, p. 25), "the net present value criterion is quite compelling, and indeed it is generally regarded as the single best measure of an investment merits". More precisely,

- if the **viability** of a single investment project is examined, the appropriate decision rule is "undertake the project if the net present value PV_0 at the required rate of return r is positive";
- if a single project has to be selected among 2 or more **mutually exclusive** investment projects, the appropriate decision rule is "undertake the project with the largest net present value $PV_0 > 0$ at the required rate of return r ". When performing such a selection, one can consider investment alternatives that differ in size and/or duration. In doing so, one tacitly assumes that the gap is filled by incremental investment projects

undertaken at the required rate of return, their present values being nought. However, this couldn't make any sense; a case in point is that of repeatable activities, dealt with in Luenberger (1998, p. 29). As return is compounded and each real investment project has the same properties as an **immediate annuity**, maximising the net present value PV_0 amounts to maximising the **net future value**

$$FV_n = \sum_{t=0}^{n} x_t (1+r)^{n-t} = PV_0 (1+r)^n$$

by virtue of **consistency**; the former is calculated on the appraisal date, the latter at the time horizon.

- Consider several **independent** investment projects, each with only one expenditure followed by many receipts; suppose that one or more projects have to be selected among them under a **budget constraint**. Even if the goal is that of finding the project combination with the largest net present value, investment projects must be ranked by their profitability index, as shown in Exercise **27**.

In our opinion, although sticking to the above-mentioned decision rules is appropriate, the computation of the internal rate of return and the payback period of each investment project might provide useful additional information.

In the following

- we will mainly focus on investments in a **strict form**, whereby all expenditures occur earlier than all receipts;
- we will take the **point of view** of a top manager rather than of an ordinary shareholder, who is not involved in the company management. Top managers raise capital, be it equity or debt, to pay the initial expenditures and use the subsequent receipts to reward stakeholders, i.e. stockholders and lenders.

Exercise 24.

Consider the following stream of **free** (**equity**) cash flows, where time is measured in years and x_t^+ (x_t^-) denotes a receipt (an expenditure).

x_0^-	x_1^-	x_2^+	x_3^+	
0	1	2	3	time

Suppose that the rate of return required by stakeholders (stockholders) is *r* per year. Show that, when calculating the net present value of this investment project, the required rate of return represents a cost of capital as well as a reinvestment rate.

Solution.

The net present value at time 0 and the net future value at time 3 are worth

$$PV_0 = x_0^- + x_1^-(1+r)^{-1} + x_2^+(1+r)^{-2} + x_3^+(1+r)^{-3}$$

$$FV_3 = (1+r)^3 PV_0 = x_0^-(1+r)^3 + x_1^-(1+r)^2 + x_2^+(1+r) + x_3^+$$

FV_3 has the same sign of and is proportional to PV_0. Therefore, PV_0 may be replaced by FV_3 as a financial indicator, the latter being an accumulation of expenditures and receipts. When shifting an expenditure (a receipt) forward in time in the latter equation, *r* represents a cost of capital (a reinvestment rate). Such a property applies to any stream with any number of cash flows, provided that return is compounded.

Exercise 25.

A property management company can

a) either keep renting a housing estate it owns for 5 more years and receive €0,6 million per year net of all the expenses. The resale value of the housing estate after 5 years is likely to be €11 million.

b) or sell the housing estate now for €10 million and effect an alternative investment at the yearly rate of return of 8%.

Determine the most favourable alternative when depreciation, inflation and taxes are disregarded.

Solution.

Note that what was paid for the housing estate is sunk cost and hence irrelevant for our analysis. Since the present values of the 2 alternatives are (time is measured in years and cash flows are expressed in 10^6 €)

$$PV_A = 0{,}6 a_{5|8\%} + \frac{11}{1{,}08^5} = 9{,}882 \quad \text{and} \quad PV_B = 10$$

the latter (sell now) is better than the former (sell after 5 years).

REMARK.

As shown in Exercise **24.**, the net future value criterion is equivalent to the net present value one owing to consistency. Since the future values after 5 years of the 2 alternatives are (time is measured in years and cash flows are expressed in 10^6 €)

$$FV_A = PV_A 1{,}08^5 = 0{,}6 s_{5|8\%} + 11 = 14{,}520 \quad \text{and} \quad FV_B = PV_B 1{,}08^5 = 10 * 1{,}08^5 = 14{,}693$$

the previous ranking is obtained again, with the latter alternative (sell now and reinvest) being better than the former (reinvest all the receipts and sell after 5 years).

Exercise 26.

A businessman is considering the purchase of some office machines worth €80.000 on either of the following terms

a) cash payment, which involves a 8% discount;

b) payment in instalments: an immediate payment of €16.000 is followed by 4 half-yearly payments, each of €16.000 and due in arrears.

Determine the most favourable terms under the conventional assumption that the businessman can borrow (and lend) money at the rate of 6,09% **per year effective**.

Solution.

The equivalent rate **per half year** is $i_2 = \sqrt{1{,}0609} - 1 = 3\%$. Since the present values of the 2 alternatives are

$$PV_a = 73.600 \text{ €} \quad \text{and} \quad PV_b = 16.000 + 16.000 a_{4|3\%} = 75.473{,}57 \text{ €}$$

the former terms are less expensive.

Exercise 27.

The management of a company could invest at most €500.000 in one or more of 5 projects in a **strict form**. In all instances an initial outlay is followed by a stream of receipts, as reported in the table below.

project	outlay (€)	PV receipts (€)
1	100.000	190.000
2	100.000	180.000
3	200.000	300.000
4	250.000	500.000
5	250.000	400.000

Each investment project can be carried out only on a full scale. Find the optimal investment combination and its net present value by a **heuristic** method.

Solution.

Let NPV denote net present value and PI denote profitability index, namely a **benefit-cost** ratio. Since $\mathrm{NPV} = \mathrm{PV\,receipts} - \mathrm{outlay}$ and $\mathrm{PI} = \dfrac{\mathrm{PV\,receipts}}{\mathrm{outlay}}$ with $\mathrm{NPV} \geq 0 \Leftrightarrow \mathrm{PI} \geq 1$, the previous table can be expanded as follows

project	outlay (€)	PV receipts (€)	NPV (€)	PI
1	100.000	190.000	90.000	1,9
2	100.000	180.000	80.000	1,8
3	200.000	300.000	100.000	1,5
4	250.000	500.000	250.000	2
5	250.000	400.000	150.000	1,6

Projects 4, 1 and 2 have the largest **PI**'s and require a total outlay of €450.000, which meets the budget constraint of €500.000. Projects 4, 1 and 2 together have the largest **NPV**, equal to €420.000. Recall that this heuristic method gives an approximate solution, quite accurate in more articulated problems. To obtain the optimal solution, a zero-one optimisation problem must be solved (Luenberger, 1998, chapt. 5).

3.3. Internal rate of return (IRR)

Let time t be measured in years, time 0 be the appraisal date and t_k be a rational number. Consider a **real investment project**, represented as a stream of payments including forecast **expenditures** $\left(x_k^- < 0\right)$ and **receipts** $\left(x_k^+ > 0\right)$, as shown in the following diagram

x_0	x_1	x_2		x_{n-1}	x_n
0	t_1	t_2	$\cdots$	t_{n-1}	t_n

The **internal rate of return** is an appropriate real root $\bar{r}$ of the equation

$$0 = \sum_{k=0}^{n} x_k (1+r)^{-t_k}$$

such that $\bar{r} > -1$. Any multiple of a real investment project has the same internal rate of return. In some instances the root $\bar{r}$ could not exist or there could be several real roots greater than -1; however, for the important class of investments in a **strict form** the real root $\bar{r}$ exists, is unique, and can take any sign. The following propositions on the internal rate of return (**per year**) could come in useful when checking a financial model for errors.

Proposition.

If all expenditures occur earlier than all receipts, the internal rate of return is well-defined. In other words, there exists a unique appropriate root, which may be positive, zero, or negative.

Proposition.

If receipts exceed expenditures $\left(\sum_{k=0}^{n} x_k > 0 \right)$ and all expenditures occur earlier than all receipts, the internal rate of return is **unique** and **positive**.

Example 14.

x_t	-5	-5	-5	10	10
t	0	1	2	3	4

Solution.

Since $\text{total receipt} = 20 > \text{total expenditure} = 15$, and all expenditures occur earlier than all receipts, the internal rate of return r is well-defined, positive and equal to 12,1% **per year**, as can be checked by using the **IRR** built-in function of a spreadsheet package.

SKETCH OF PROOF.

Consider the above stream of payments and let PV_t, FV_t and V_t denote its present value, future value and value at time *t*. The value at time 2 of this stream of payments is

$$V_2 = FV_2 + PV_2 = -5\left[(1+r)^2 + (1+r) + 1\right] + 10\left[(1+r)^{-1} + (1+r)^{-2}\right]$$

Consider V_2 as a function of *r*. Since receipts exceed expenditures, we have $V_2(0) = -15 + 20 = +5$. Moreover, as *r* increases, $V_2(r)$ decreases (both FV_2 and PV_2 do so) with $\lim_{r \to +\infty} V_2(r) = -\infty$. Since $V_2(r)$ is a continuous function, it must cross the *r* axis once and only once at some positive value $\bar{r}$, which is the unique internal rate of return. Indeed, $V_2(\bar{r}) = 0$ implies that

$$PV_0(\bar{r}) = (1+\bar{r})^{-2} V_2(\bar{r}) = 0$$

owing to consistency. It is readily realised that a similar proof carries over to any stream of payments meeting the assumptions.

REMARK.

The **average due date** of a sequence of payments is a weighted average of all payment dates, the weights being the ratios between each payment and the sum of all payments. As proved by the Italian mathematician Eugenio Levi (1913-1969), if receipts exceed expenditures $\left(\sum_{k=0}^{n} x_k > 0 \right)$ and the average due date of all expenditures falls earlier than the payment date of the first receipt, there exists only one **positive** internal rate of return. Notice that the proposition doesn't rule out the existence of other roots, which must be negative; a case in point is provided by Example **15**. When the above assumption on the average due date is met, the first payment x_0 is an expenditure.

Example 15.

x_t	-20	-20	15	15	15	15	-10
t	0	1	2	3	4	5	6

Solution.

Since $\text{total receipt} = 60 > \text{total expenditure} = 50$, and the **average due date** of all expenditures is $\frac{0*20 + 1*20 + 6*10}{20 + 20 + 10} = 1{,}6 < 2$, there is only one positive internal rate of return. Moreover, we have $\mathrm{PV}_0(0) = \sum_{k=0}^{6} x_k > 0$ and $\lim_{r \to -1^+} PV_0(r) = -\infty$, as the last payment is an expenditure and $-10(1+r)^{-6}$ is the dominant term of $PV_0(r)$ for $r \to -1^+$. Therefore, there must be at least a negative root. Indeed, there are 2 internal rates of return, respectively equal to -58,4% and 9,3%

per year, as can be checked by appropriately using the **IRR** built-in function of a spreadsheet package.

Proposition (C.J. Norstrøm, 1972).

Let payment dates be **evenly spaced** and $B_t = x_0 + x_1 + \ldots + x_t$ be the cash balance at time t, after the payment x_t has occured. If $B_t \neq 0$ for $t = 0, 1, \ldots, n$ and the sequence $\{B_0, B_1, \ldots, B_n\}$ contains precisely one change of sign, then the present value equation has a **unique positive root** (keep in mind that if $x_t \neq 0$, $B_t = 0$ must be considered as a change of sign).

Example 16.

x_t	-5	1	-3	8	4
B_t	-5	-4	-7	1	5
t	0	1	2	3	4

Solution.

As shown in the second row of the above diagram, there is only one change of sign in the time pattern of the cash balance, that takes place between time 2 and 3. Therefore, the internal rate of return r is well-defined, positive and equal to 22,1% **per year**, as can be checked by using the **IRR** built-in function of a spreadsheet package.

The following properties of $PV_0(r)$, the net present value at time 0 of a real investment project in a **strict form**, will come useful in the sequel; they are illustrated by the figures of Exercise **28** and Exercise **29**.

Proposition.

If all expenditures occur earlier than all receipts, we have $PV_0(r) > 0$ only for $-1 < r < \bar{r}$, where $\bar{r}$ is the **unique** internal rate of return. Indeed, the net present value $PV_0(r)$, whenever **positive**, is a **decreasing** and **convex** function of the required rate of return r with $\lim_{r \to -1^+} PV_0(r) = +\infty$.

Moreover, $PV_0(r)$ can have only one stationary point, which is a negative minimum.

PROOF.

As *r* converges to -1^{+}, $x_n^{+}(1+r)^{-t_n}$ is the dominant term of $PV_0(r)$ so that the limit tends to positive infinity. Let time τ fall between the payment date of the last expenditure and the payment date of the first receipt. The first derivative of $PV_0(r)$ with respect to *r* is such that

$$\frac{dPV_0(r)}{dr} = \sum_{k=0}^{n} x_k(-t_k)(1+r)^{-t_k-1} < \sum_{k=0}^{n} x_k(-\tau)(1+r)^{-t_k-1} = -\tau(1+r)^{-1}PV_0(r)$$

whereas the second derivative of $PV_0(r)$ with respect to *r* is such that

$$\frac{d^2PV_0(r)}{dr^2} = \sum_{k=0}^{n} x_k t_k(t_k+1)(1+r)^{-t_k-2} > \sum_{k=0}^{n} x_k t_k(\tau+1)(1+r)^{-t_k-2} = -(\tau+1)(1+r)^{-1}\frac{dPV_0(r)}{dr}$$

Therefore, $PV_0(r) > 0$ implies that $\frac{dPV_0(r)}{dr} < 0$ and $\frac{d^2PV_0(r)}{dr^2} > 0$. Moreover, $\frac{dPV_0(r)}{dr} = 0$ implies that $\frac{d^2PV_0(r)}{dr^2} > 0$; in other words, each stationary point is a negative minimum. As two minima are separated by a maximum, there can be only one stationary point.

Joint use of the NPV and IRR

Although reference is often made to the internal rate of return in the business place, the internal rate of return (IRR) is more of a complement to rather than a substitute of the net present value (NPV). Keep in mind that the IRR is both a cost of capital and a reinvestment rate; therefore, it becomes meaningless when very large, since receipts cannot actually be reinvested at the calculated rate. As previously mentioned, a wary decision maker should, in our opinion, take both financial indicators into consideration, as they provide useful information; nonetheless, he/she should stick to the NPV whenever the two attendant decision criteria are in contrast.

IRR and NPV prove consistent and give an identical answer in two important instances. This follows from the previous propositions. If the **viability** of a single investment project in a **strict form** is examined, the rule "undertake the project if the internal rate of return $\bar{r}$ is greater than the required rate of return *r*" is equivalent to the rule "undertake the project if the net present value PV_0 at the required rate of return *r* is positive".

SKETCH OF PROOF.

As previously proved, we have $PV_0(r)>0$ only for $-1<r<\bar{r}$, where $\bar{r}$ is the **unique** internal rate of return. There are 2 possibilities:

1) if receipts do not exceed expenditures $\sum_{k=0}^{n} x_k \leq 0$, the unique internal rate of return $\bar{r}$ is nonpositive so that $PV_0(r)<0$ for all $r>0$. The project is **not viable**: the internal rate of return is lower than any positive required rate of return so that the net present value is negative at any required rate of return;

2) if receipts do exceed expenditures $\sum_{k=0}^{n} x_k > 0$, the unique internal rate of return $\bar{r}$ is positive so that $PV_0(r)>0$ for $0 \leq r<\bar{r}$. The project is **viable** if the internal rate of return is larger than the required rate of return so that the net present value is positive at the required rate of return.

If a single investment project has to be selected among 2 or more **mutually exclusive** investment projects in a **strict form**, use can be made of the incremental IRR's relative to suitable incremental projects, as explained in Exercise **29**. However,

1) reference is made to the incremental IRR's rather than the actual ones;
2) the notion of incremental project could prove somewhat abstract;
3) the selection procedure could fail.

Exercise 28.

Consider the following real investment project in a **strict form**, where time is measured in years and an outlay x_0^- is followed by several receipts x_k^+ with $k=1,2,\ldots,n$.

x_0^-	x_1^+	x_2^+		x_n^+	
0	1	2	$\cdots$	n	time

The net present value at time 0 of the investment project is

$$PV_0 = x_0^- + \sum_{k=1}^{n} x_k^+ (1+r)^{-k}$$

where r is the required rate of return per year.

a) Suppose that receipts are larger than the outlay

$$x_0^- + \sum_{k=1}^{n} x_k^+ > 0$$

find the qualitative features of the graph of $PV_0(r)$ (hint: recall that, as proved in Section 3.3, $PV_0(r) > 0$ for $-1 < r < \bar{r}$).

b) Let $n = 3$, $x_0^- = -1.400$ and $x_t^+ = 550$; cash flows are expressed in 10^3 €. Check that the internal rate of return is $\bar{r} = 8{,}688\%$.

Solution.

Since receipts are larger than the initial outlay by assumption and the initial outlay occurs earlier than all receipts, the internal rate of return is unique and positive, as proved in Section 3.3 and shown in the drawing below.

a) When an **initial outlay** is followed by **several receipts**, we have

1) $PV_0(0) = x_0^- + \sum_{k=1}^{n} x_k^+$;

2) $\lim_{r \to +\infty} PV_0(r) = x_0^-$ so that the initial outlay is a **horizontal asymptote**;

3) $\frac{dPV_0(r)}{dr} = \sum_{k=1}^{n} x_k^+ (-k)(1+r)^{-k-1} < 0$ so that $PV_0(r)$ **slopes downward**;

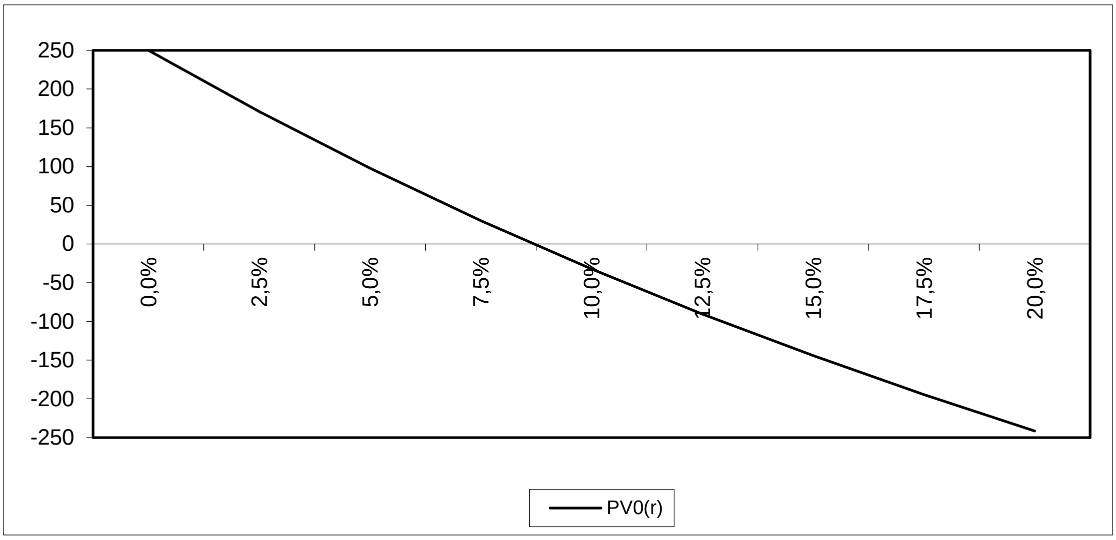
4) $\frac{d^2 PV_0(r)}{dr^2} = \sum_{k=1}^{n} x_k^+ k(k+1)(1+r)^{-k-2} > 0$ so that $PV_0(r)$ has a **convex shape**.

Since $PV_0(r)$ is a **continuous** function, $PV_0(0) > 0$ by assumption, and $\lim_{r \to +\infty} PV_0(r) < 0$, the horizontal axis is crossed at some **positive** value $\bar{r}$, the unique **internal rate of return**.

b) We have

$$PV_0(8{,}688\%) = -1.400 + 550 a_{3|8{,}688\%} = 0$$

as implied by the definition of internal rate of return.



Exercise 29.

The management of a company is considering whether to undertake either of the following real investment projects in a **strict form** (time is measured in years and cash flows are expressed in 10^3 €).

a) For each investment project draw the graph of the net present value $PV(r)$ at time 0 as a function of the required rate of return *r*.

b) Suppose that the required rate of return is 8% and determine what project should be funded.

A	-10	10	1	1
B	-10	1	1	12
time	0	1	2	3

Solution.

The NPV's of both projects as functions of *r* are plotted in the drawing below.

a) In both instances,

1) as one **outlay** is followed by **several receipts**, the net present value $PV(r)$ is a decreasing and convex function of *r*, with the outlay being a horizontal asymptote

$$\lim_{r\to+\infty} PV_A(r) = \lim_{r\to+\infty} PV_B(r) = -10.000$$

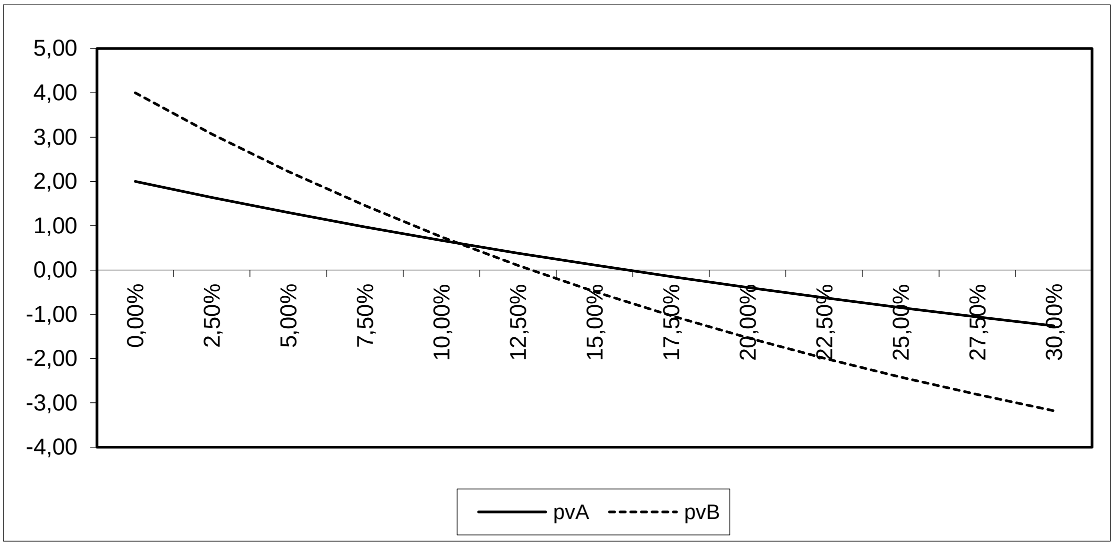
2) as receipts are larger than the outlay ($PV_A(0) = 2.000$ and $PV_B(0) = 4.000$), the internal rate of return *IRR* is unique and positive.

Moreover, the 2 graphs have only one common point

$$PV_A = PV_B = 603{,}60 \text{ for } r = 10{,}554\%$$

whereas $PV_A(20\%) = -393{,}52$ and $PV_B(20\%) = -1.527{,}78$; therefore, we have

$$PV_A < PV_B \text{ for } r < 10{,}554\% \text{ and } PV_A > PV_B \text{ for } r > 10{,}554\%$$



b) Although $IRR_A = 16{,}044\% > IRR_B = 12{,}937\%$, as can be numerically ascertained, project B is more profitable, since receipts are likely to be reinvested at a 8% rate

$$PV_A(8\%) = 910{,}43 \text{ €} < PV_B(8\%) = 1.309{,}25 \text{ €}$$

REMARK.

When two **mutually exclusive** investment projects *A* and *B* are considered, the *NPV* and *IRR* decision criteria may be reconciled to some extent so as to give an identical solution. This happens, if project A, project B and a suitable incremental project are in a **strict form**, i.e. they have each all expenditures that occur earlier than all receipts so that the function $PV(r)$ of each project decreases and is convex whenever $PV(r) > 0$. The *IRR* criterion can then be used as follows

- first check that the 2 projects under examination meet the previous assumption;

- then consider the investment project with the smallest net receipt, which in our case is project A. As $r = 8\% < 16{,}044\% = IRR_A$, project A could be undertaken; notice that $PV_A(r) > 0$ for $-1 < r < 0{,}16044$ so that $PV_A(0{,}08) > 0$;
- finally consider the incremental investment project $B - A$ with cash flows (expressed in 10^3 €)

B – A	0	-9	0	11
time	0	1	2	3

Note that all expenditures of project $B - A$ occur earlier than all receipts; if this assumption were not met, our procedure would not be applicable. As $r = 8\% < 10{,}554\% = IRR_{B-A}$, the incremental investment project B-A must be undertaken; notice that $PV_{B-A}(r) > 0$ for $-1 < r < 0{,}10554$ so that $PV_{B-A}(0{,}08) > 0$ and hence $PV_B(0{,}08) > PV_A(0{,}08)$. Since $A + B - A = B$, the investment project B must be undertaken, which is the same answer as previously.

Keep in mind that

- if project A had been rejected, step 3 of the above procedure should have been skipped and project B should have been appraised as a single investment project;
- if we had had $IRR_{B-A} < r = 8\%$, project A would have been optimal.

3.4. Adjusted present value (APV)

The evaluation procedure outlined in this section is based on the notion of **adjusted present value** and can comply with the assumption of irrelevance of the financial structure, provided that both corporate and personal taxes are taken into account. It is suitably adapted from Benninga-Sarig (1997, chap. 8) by introducing a well received rule of thumb, which minimises the risk of bankruptcy. Remarkably, the APV is both theoretically sound and simple to use, even for a nonspecialist.

Let time *t* be measured in years and 0 be the appraisal date. Consider a **real investment project**, represented as a stream of **free cash flows** including forecast **expenditures** $\left(x_k^- < 0\right)$ and **receipts** $\left(x_k^+ > 0\right)$, as shown in the following diagram

	x_1	x_2		x_{n-1}	x_n
0	t_1	t_2	$\cdots$	t_{n-1}	t_n

where n is the time horizon, and the last amount x_n is the sum of a free cash flow and a **terminal value**.

To appraise such a real investment project, we have to

- find a suitable (initial) financial gearing by trial and error. According to the accounting scheme of Section 3.1, whenever a loan is taken out, less cash is used and/or less equity is raised but comparatively lower net incomes follow. However, we must keep in mind that too high a financial gearing would make the firm too vulnerable to financial distress, bringing about too high a risk of bankruptcy. In order to ascertain whether a given (initial) financial gearing is sustainable, we must consider the case of full debt repayment by constant instalments in, say, 10 years at most and make use of a well received rule of thumb. More precisely, we must check whether the **annual debt service cover ratio**

$$\frac{\text{cash from operations}_t + \text{interest}_t}{\text{constant instalment}}$$

  lies in the interval $[1{,}3;\,2]$ for $t = 1;2;\cdots;n$, with cash from operations, an entry of the cash flow statement, being equal to net income plus depreciation minus the change in net working capital;

- simulate the financial statement for n years in a row, compute the **free cash flows** $\{x_1; x_2; \cdots; x_n\}$, and then suppose that the real investment project is ungeared. As a consequence, **free cash flows** are the same as **equity cash flows**. However, as financial gearing is introduced, the latter change, whereas the former generally do not. Let r^* be the required rate of return, i.e. the appropriate cost of equity for an ungeared firm picked out from Table 1. The net present value of the all equity financed project is

$$PV_0^* = \sum_{t=1}^{n} x_t \left(1 + r^*\right)^{-t}$$

- consider the effects of financial gearing. The downside of borrowing is the added risk of bankruptcy, which is negligible in our case, whereas the upside of borrowing is the tax shield provided by deductible interest. As the net present value is a linear operator with

free cash flows being generally the same as the original **equity cash flows**, we can calculate the **adjusted present value** PV_0 of the real investment project as the sum of 2 terms

$$PV_0 = PV_0^* + PV_0^{**}$$

with PV_0^* being the net present value of the all equity financed project and PV_0^{**} being the net present value of debt. The appropriate cost of equity r^* for an ungeared firm is used to calculate the former, whereas the cost of debt *i* is used to calculate the latter; in principle, we have $r^* > i$. It is readily ascertained that the latter is equal to the net present value of the interest tax shield, i.e.

$$PV_0^{**} = \sum_{t=1}^{n} \text{interest}_t \, \tau (1+i)^{-t} + TS_0$$

where τ is the corporate tax rate and TS_0 is the present value at time 0 of the interest tax shield beyond time *n*. Indeed, each amount borrowed is equal to the present value at the rate *i* of all subsequent interest and capital repayments. Whenever the risk of bankruptcy is not negligible, the present value of the expected bankruptcy costs must be taken away from PV_0^{**}. According to the usual approach, the present value of the expected bankruptcy costs is equal to the probability of bankruptcy times the present value of the bankruptcy costs. The estimation process is indirect and hard. More precisely, Table 4 is a typical source for the probability of bankruptcy, provided that a bond rating has been estimated. Moreover, studies on actual bankruptcies are the source for the present value of the bankruptcy costs; however, considerable errors are very likely to occur. This is unfortunately a weakness of the adjusted present value.

REMARK.

Actually, as **modelling errors** may occur, the simulated free cash flows are not completely reliable. In **engineering sciences**, modelling errors are tackled by introducing an adequate **safety margin**; accordingly, the adjusted present value PV_0 ought to be adequately greater than 0 and each annual debt service cover ratio ought to be adequately greater than 1,3.

REMARK.

Let D_0 be the market value of net debt at time 0, where

net debt =debt–cash–bank balances–marketable securities–credit other than receivables

If the real investment project is to be undertaken by a start up, the adjusted present value PV_0 is equal to its **market value** of **total capital**, also known as **enterprise value**, whereas $PV_0 - D_0$ is equal to its **market value** of **equity capital**. If the real investment project is to be undertaken by an existing firm, such market values are **incremental**.

REMARK.

Whenever the debt to equity ratio changes considerably over time, the **adjusted present value** is a more appropriate tool than the **weighted average cost of capital**. A leveraged buy out is a case in point. As shown in Benninga-Sarig (1997, chapt. 8), the weighted average cost of capital r_{WACC} takes the form

$$r_{WACC} = wr^{**} + (1-w)i(1-\tau)$$

where r^{**} is the appropriate cost of equity for a geared firm, $i(1-\tau)$ is the after tax cost of debt, the weight w is the ratio between the market value of equity capital and the enterprise value, and the weight $(1-w)$ is the ratio between the market value of debt and the enterprise value.

It is usually supposed that r^{**} lies on the **security market line**, a construct of the **capital asset pricing model** presented in Part II.

Unfortunately, the weighted average cost of capital suffers from 2 serious shortcomings. First, the market value of equity capital and hence the enterprise value are usually unknown, as they are the outputs of the evaluation procedure. As a consequence, market values are often replaced with book values. Moreover, gearing, as measured by the weight $(1-w)$, is assumed to be constant with time, which is unlikely to be the case in practice.

REMARK.

According to the Italian tax law, **interest payable**, less **interest receivable**, is allowable against taxable income, in general by an amount not greater than 30% of **EBITDA**. Any excess may be deducted subsequently, provided that the 30% constraint is met; the unused EBITDA may be employed subsequently as well. The provision does not apply, among others, to banks, insurance companies, and state-controlled companies that supply water, energy, district heating, or dispose of waste, or purificate.

We can also solve an inverse problem, in which the adjusted present value PV_0 is assigned and the cost of total capital *r* is to be determined. The cost of total capital sought makes the net present value of all free cash flows equal to $PV_0 = PV_0^* + PV_0^{**} = \sum_{t=1}^{n} x_t (1+r)^{-t}$. Such an equation can have multiple roots.

Proposition.
If the free cash flows $\{x_1; x_2; \cdots; x_n\}$ are such that all stakeholders' expenditures occur earlier than all stakeholders' receipts, there exists a unique cost of capital *r*, which is lower than the cost of equity r^* for an ungeared firm, i.e. $r < r^*$.

PROOF.
If PV_0 is treated as a fictitious initial expenditure, computing *r* amounts to computing an internal rate of return. As all expenditures occur earlier than all receipts, the internal rate of return is well defined. Moreover, $\sum_{t=1}^{n} x_t (1+r)^{-t}$ is, whenever positive, a decreasing and convex function of *r* with $\lim_{r \to -1^+} PV_0(r) = +\infty$. As

$$PV_0 = \sum_{t=1}^{n} x_t (1+r)^{-t} > PV_0^* = \sum_{t=1}^{n} x_t \left(1+r^*\right)^{-t}$$

it follows that $r < r^*$.

Example 17.
Reference is made to an incomplete feasibility study carried out at Politecnico di Milano in 2007. The aim was to ascertain whether a **biomass combined heat and power station** could be located in a plain farming area in Northern Italy. Such a station operates on wood, wood industry waste, pruned branches, cereals, and residues of cereals (e.g. leaves, awns, stems), so producing **renewable energy**, i.e. MW1,1 of **green electricity** and MW12 of **heat**, supplied within a maximum reach of km15 through **district heating**. It has a high operational efficiency (=output energy/input energy) of 80% at most in the colder months, compared with 35% at most

for a conventional plant that supplies only electricity. It has no net release of CO_2 : the carbon dioxide released into the atmosphere was previously absorbed from it during the recent biomass growth.

The business is to be undertaken by a new state-owned firm. More specifically, 3 years are needed to design the power station and get the planning permission from town hall; 1 year is needed to build the power station, which has a life cycle of 30 years. Therefore, free cash flows are negative for the first 4 years of the real investment project and positive for the subsequent 30 operating years, in which capacity is not expanded and renewal investment is small. Equity is only raised and debt is only taken out during the first 4 years. The debt to equity ratio is constant over time and equal to 1,5; needless to say book values are considered. Such a financial gearing is sustainable, as it is readily ascertained that all debt can be safely repaid over 8 operating years. For simplicity's sake, no dividend is paid during the 30 operating years, with retained earnings being turned into cash, which can be used to repay debt. Green electricity generation is subsidized for the first 8 operating years.

A *pro-forma* financial statement is projected for $n = 14$ years in a row and an enterprise value-EBITDA ratio equal to 7,5 is used to compute the terminal value (see Section 3.5). The bottom line of such a simulation is the sequence of $n = 14$ **free cash flows** (expressed in 10^3 €) reported in the following table

x_t	-2.078,0	-2.161,1	-2.249,5	-4.421,8	1.670,3	1.959,0	1.959,0
t	1	2	3	4	5	6	7

x_t	1.959,0	1.959,0	1.959,0	1.959,0	1.959,0	963,2	11.390,4
t	8	9	10	11	12	13	14

The appropriate cost of equity for an ungeared firm is $r^* = 8\%$. The cost of debt is $i = 6{,}5\%$ and the net present value of the interest tax shield is $PV_0^{**} = 1.504{,}63$.

Find the adjusted present value PV_0 and the cost of total capital *r*.

Solution.

Discounting the sequence of free cash flows at the rate $r^* = 8\%$ obtains $PV_0^* = 3.497{,}61$. Therefore, we have

$$PV_0 = PV_0^* + PV_0^{**} = 3.497{,}61 + 1.504{,}63 = 5.002{,}24$$

such an adjusted present value being an estimate of the enterprise value at time 0. Computing *r* amounts to computing the internal rate of return of the following sequence of cash flows

	-7.080,2	-2.161,1	-2.249,5	-4.421,8	1.670,3	1.959,0	1.959,0
t	1	2	3	4	5	6	7

	1.959,0	1.959,0	1.959,0	1.959,0	1.959,0	963,2	11.390,4
t	8	9	10	11	12	13	14

Using the IRR built-in function of a spreadsheet package obtains $r = 6{,}8\%$.

3.5. Appraisal of a company in business practice

Discounted cash flows and **financial multiples** are the 2 main appraisal methods used in business practice. Many Anglo-Saxon professors of Corporate Finance are in favour of discounting cash flows, whereas many market practitioners mostly rely on financial multiples. In principle, both methods may be used and reconciled with each other.

In the former case, appraisal is **analytical** and **prospective**. A balance sheet and an income statement are projected (on a spreadsheet) for 3-5 or 10 subsequent years so that a cash flow statement and a sequence of **free cash flows** can be obtained. Recall that only business activities are considered, whereas the management of excess cash, a financial activity, is disregarded. Moreover, a **terminal enterprise value** after 3-5 or 10 years is calculated, by the use of a dividend discount model and/or of a financial multiple; an estimate of the **enterprise value** is then obtained as the present value of free cash flows and terminal value, both discounted at a **cost of total capital**, often a weighted average cost of capital (see Benninga and Sarig, 1997, chapts. 3 and 8). Alternatively, an adjusted present value can be calculated by applying the theoretically sounder procedure presented in Section 3.4. Finally, **equity cash flows** can also be discounted at a **cost of equity capital**, the outcome being an estimate of the **market value of equity capital** (see Benninga and Sarig, 1997, chapt. 13). Free cash flows are hard to use

whenever marking loans to market is hard. In constrast, equity cash flows are hard to use whenever convertible bonds and warrants are involved.
In the latter case, appraisal is **empirical** and may be **retrospective**, as it usually occurs in the field of private equity. The enterprise value may be appraised through either of the equations:

$$\text{enterprise value} = k_{EBIT}\ \text{EBIT}$$

$$\text{enterprise value} = k_{EBITDA}\ \text{EBITDA}$$

where k is the financial multiple and EBIT(DA) is either a 3-5 year historical average or the latest available figure of earnings before interest, taxes (and depreciation), namely of the net (gross) operating income. Actually, reference can also be made to the proceeds of sales. Keep in mind that proper adjustments are made to the book figure of net or gross operating income. The multiple k is a suitable **average** of the multiples attached to various listed comparable companies operating in the business sector under scrutiny. In general, multiples can be either **trailing** or **leading** according to whether use is made of either past data or forecasts about the future. Leading multiples are more appealing in theory; trailing ones are easier to compute and less subjective as well.

REMARK.
Suppose that the target of a **leveraged buy out** is an Italian manufacturing company with a consolidated position in a **mature** business sector. According to a reliable heuristic rule, its **operating ratios** should be such that

EBITDA/sales=12-13%, working capital/sales=30-35%, capital expenditures/sales=1-2%

where capital expenditures are asset renewal expenditures. Although profitability is not high, which is in line with a mature business sector, working capital and capital expenditures are in a favourable proportion so that a mild increase in sales wouldn't be harmful. In principle, as long as

EBIT/interest>2 and debt/EBITDA<4,5

additional debt can be obtained from a bank.

Italian experts make use of the **asset-based** method as well, which is **analytical** and **retrospective**. More precisely, they appraise the market value of equity by adjusting the entries of the most recent balance sheet under the conventional assumption that the business is being **liquidated**, i.e. is not a **going concern** any longer. Disparities between market and book values are likely to involve, e.g., intangible assets, participated companies owing to the consolidation method, fixed assets owing to inflation and the gap between fiscal and actual depreciation, credits and debts that don’t take the form of a listed security (for instance, the book value of a credit payable 2 years from now is the face value, whereas the market value is its present value). The asset-based evaluation is consistent with the peculiar case in which the average profitabilty of the company is normal, i.e it meets investors’ requirements precisely. This explains why the asset-based method is ancillary and shouldn’t be used alone.

4. Fixed income securities and basics of bond management

4.1. Appraisal of fixed rate bonds with yearly coupons

Let time be measured in **years**. Consider a **fixed rate** bond with **yearly** coupons and face value of 100 **percent**; let c be the **yearly** coupon rate and n be the number of remaining coupons. As shown by the following diagram, the bond pays a coupon $100c$ at the end of **each year** and the face value 100 at the **maturity** n. As a consequence, there is no **redemption premium**, nor does the issuer benefit from an embedded call option, i.e. from the right to repay the bond prior to the maturity date.

		$100c$	$100c$		$100c$	$100(1+c)$
0	t	1	2	$\cdots$	$n-1$	n

Moreover, if the latest coupon payment (or the bond issue) took place at time 0, $0 \le t < 1$ is the time passed since then so that the **accrued interest** is worth $100ct$. When not stated otherwise, **settlement lags**, **commissions**, **fees**, and **taxes** are assumed away and 30-day months are considered throughout this section, in line with the 30/360 European day count convention.

Suppose that the bond price is publicly available and reported, e.g., in the database of an information provider and in a page of a financial newspaper. Although such bond quote is a **clean price** P_{clean}, any buyer of the bond must pay a **dirty price** P_{dirty} to the seller, where $P_{dirty} = P_{clean} + 100ct$, with both prices being expressed as a **percentage** of the actual face value of the bond.

Maturity	Coupon rate	Clean price	Yield to maturity
1/2/2009	3,00%	100,050	2,61%
15/4/2009	3,00%	100,170	2,53%
1/5/2009	4,50%	100,780	2,56%
15/6/2009	3,75%	100,680	2,47%
1/11/2009	4,25%	101,460	2,63%

Table 2 – BTP quotes on 27.11.2008; coupons are half-yearly
(adapted from: il Sole 24 Ore, Friday 28.11.2008)

Example 18.

On Monday February 26th an individual investor buys some corporate bonds with face value of €75.000, coupon rate of 6%, and 5 remaining yearly coupons. The latest payment date was Thursday November 2nd. Suppose that the clean price is 104,58. Find the dirty price and the invoice price under the assumption that each month has 30 days.

Solution.

The latest coupon was paid on Thursday November 2nd so that

$$t = \frac{114}{360}$$

as $28+30+30+26=114$ days have already gone by. Therefore, the accrued interest and dirty price are

$$100 * 0{,}06\frac{114}{360} = 1{,}90 \quad \text{and} \quad P_{dirty} = P_{clean} + 1{,}90 = 104{,}58 + 1{,}90 = 106{,}48$$

Since the dirty price is expressed as a percentage of face value, the dirty price of 1 unit of face value is $\frac{106{,}48}{100}$, whereas the invoice price of 75.000 units of face value is $75.000\frac{106{,}48}{100} = 79.860$ € .

Settlement lags, **commissions**, and **taxes** are explicitly considered in Exercise **30** and Exercise **32**. Let t be the present time $(0 \le t < 1)$; suppose that a bond is bought at time t and held until expiry. The bond's gross **yield to maturity** y per year is the **internal rate of return** defined by the equation

$$P_{dirty} = P_{clean} + 100ct = \left(100ca_{n|y} + 100(1+y)^{-n}\right)(1+y)^{t}$$

under the assumption that interest is **compounded yearly**. Its right-hand side can be derived in accordance with the notion of **consistency**: all subsequent payments are discounted to time 0 and then their present value is accumulated until time *t*. The initial expenditure P_{dirty} is thus followed by *n* receipts, with their total being $n100c+100>P_{dirty}$, since bond traders are rational agents. Owing to the sufficient condition of Section 3.3 the internal rate of return is **unique** and **positive**. This equation is mentioned as Makeham's formula in McCutcheon-Scott (1986, p. 156). For $t=0$ it is the same as equation (3.2) with $m=1$ (Luenberger, 1998); for $0<t<1$ it extends equation (3.2) with $m=1$.

REMARK.

The previous equation in the unknown *y* has no closed-form solution. Nonetheless, a **numerical** (**approximate**) solution can be obtained by using a spreadsheet package like Excel. A starting point for the iterative numerical procedure is given by the approximate formula

$$\hat{y}=\frac{100c+\left(100-P_{clean}\right)/n}{\left(100+2P_{clean}\right)/3}$$

Our setting is **semideterministic**, as both **interest rate** and **credit risk** are taken into account. The actual yield on a bond is **unknown** and usually other than the yield to maturity; all payments are **semicertain**.

Adjustment to half-yearly (quarterly) coupons

The previous equation applies also to bonds with **half-yearly** (**quarterly**) coupons. In that instance, however, time must be measured in **half years** (**quarters**) and both the coupon rate *c* and the yield to maturity *y* must be expressed on a **half-yearly** (**quarterly**) basis.

Yield to maturity-price relation

Consider a **fixed rate** bond with face value of 100 percent and *n* **yearly** coupons equal to 100*c*. Let time be measured in **years** and time 0 be the present time so that the accrued interest is nought and dirty price and clean price are the same

$$P_{dirty}=P_{clean}=100ca_{n|y}+100(1+y)^{-n}$$

Consider the clean price $P_{clean}(y)$ as a function of the yield to maturity *y* per year. We have

1) $P_{clean}(0) = n100c + 100 = (nc+1)100 > 100$;

2) $P_{clean}(c) = 100c\dfrac{1-(1+c)^{-n}}{c} + 100(1+c)^{-n} = 100$;

3) $\lim_{y \to +\infty} P_{clean}(y) = 0^{+}$ so that the *y* axis is a **horizontal asymptote**;

4) $\dfrac{dP_{clean}(y)}{dy} = -n(1+y)^{-n-1}100 + \sum_{t=1}^{n} -t(1+y)^{-t-1}100c < 0$ so that $P_{clean}(y)$ **slopes downward**;

5) $\dfrac{d^2P_{clean}(y)}{dy^2} > 0$ so that $P_{clean}(y)$ has a **convex shape**.

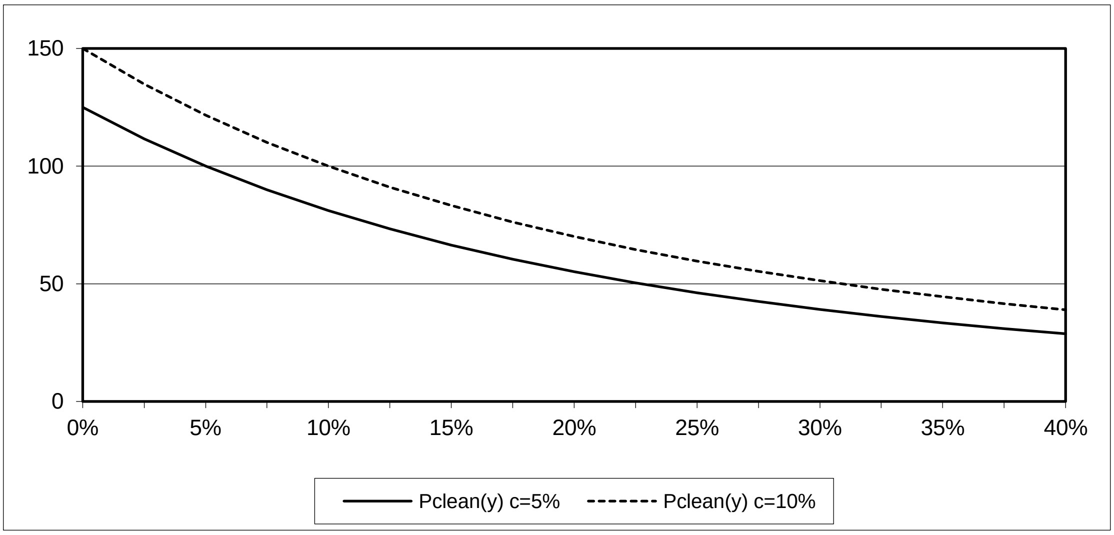
As shown in the drawing below, where $c = 5\%$ or 10% and $n = 5$, since $P_{clean}(y)$ is a **continuous** function, we have that for any given term *n* and coupon rate *c*

1) $P_{clean}(y) > 100$ for $0 \le y < c$, i.e. the coupon bond trades **at premium**;

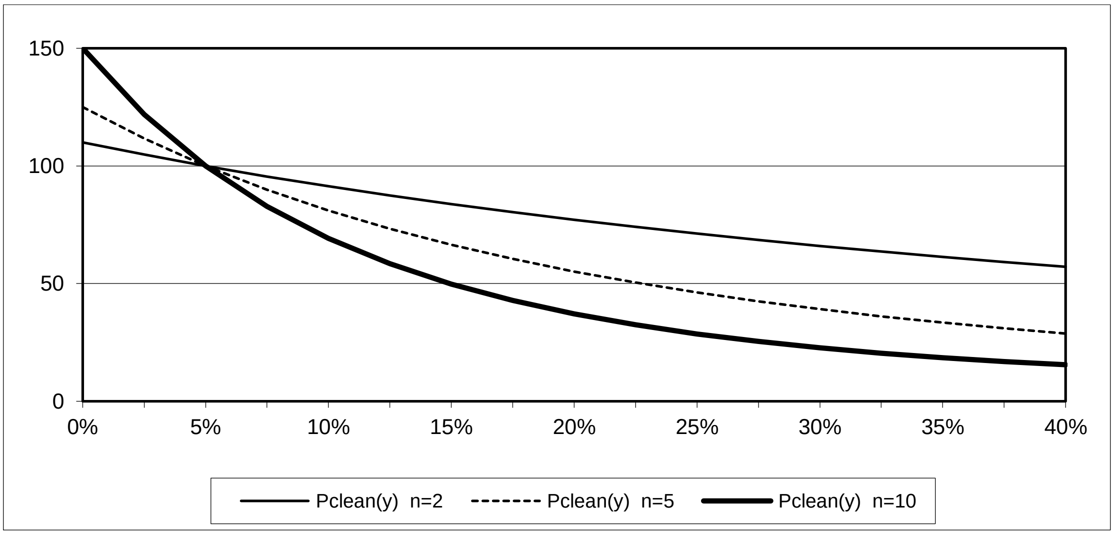
2) $P_{clean}(c) = 100$, i.e. the coupon bond trades **at par**;

3) $P_{clean}(y) < 100$ for $c < y$, i.e. the coupon bond trades **at discount**.

Moreover, for any given term *n* and yield to maturity *y*, the higher the coupon rate *c*, the higher the clean price $P_{clean}(y)$ is.



Fixed rate bonds are usually issued near par, i.e. with a coupon rate *c* close to the yearly yield to maturity *y* then required by the market for the maturity *n* and the issuer's credit rating. However, since market conditions change as time goes by, the clean price of fixed rate bonds can become other than its face value of 100, sometimes to a considerable extent. More precisely, an increase (decrease) Δy in the yield to maturity brings down (puts up) the dirty price and hence the clean price $P_{clean}(y)$. The price reaction to any given Δy is asymmetric: a positive (negative) Δy causes $P_{clean}(y)$ to drop less (rise more).



For any given yield to maturity $y < c$, it can be proved by **mathematical induction** that the further the maturity *n*, the higher the clean price is. In other words, the price-yield curve becomes steeper and turns clockwise around the par point, as the maturity *n* gets more distant. A case in point is portrayed in the drawing above, where $c = 5\%$ and $n = 2$ or 5 or 10.

Actual yield

Suppose that a **fixed rate** bond is bought at time *t* and held until expiry. Since its yield to maturity *y* is an internal rate of return, all of the coupons are implicitly assumed to be reinvested at that yield, which is unlikely to be the case. Therefore, the **actual yield** on the bond is usually other than its yield to maturity *y*. The effect exerted by the reinvestment rate is examined in Exercise **34**, where an intermediate maturity *n* is considered and reinvestment rates are consistent with a low inflation setting.

Although medium and long run reinvestment rates are very difficult to predict, prediction errors are likely to be similar for bonds with the same maturity, which might explain why, when comparing bonds with similar maturities (and similar credit ratings), bond fund managers make reference to their yields to maturity. Moreover, as explained in Gibson (2008, chapt. 13), when considering the relative appeal of US stocks and bonds in a **tactical asset allocation**, the forecast returns of the latter are simply their yields to maturity.

REMARK.

The yield to maturity of a bond portfolio is **not** a weighted average of the yields to maturity of the constituent bonds, the weights being the proportions invested in the constituent bonds.

Now consider the following peculiar financial transaction, concerning a fixed rate bond with yearly coupons, each worth 100*c*. The security is bought and resold after one year immediately after a coupon payment, the yield to maturity *y* per year remaining the same. As the bond is, in those circumstances, an annuity with *n* remaining yearly non constant payments in arrears, *y* is the **actual yield** as well. Moreover, the yearly change in the clean price is

$$\Delta P_{clean} = 100(y-c)(1+y)^{-n}$$

where *n* denotes the number of remaining coupons at purchase. We have

- $\Delta P_{clean}(y) < 0$ for $0 \leq y < c$. Although the bond trades **at premium**, the coupon received is too fat and hence offset by a capital loss;
- $\Delta P_{clean}(c) = 0$, i.e. the bond trades **at par** twice;
- $\Delta P_{clean}(y) > 0$ for $c < y$. Although the bond trades **at discount**, the coupon received is too slim and hence offset by a capital gain.

REMARK.

If *y* remains the same, each subsequent coupon payment will be accompanied by a similar occurrence. More precisely, for $y \neq c$ the clean price will get closer to 100 year after year, becoming equal to 100 at expiry; said changes are due only to the **passage of time**.

Exercise 30.

On Wednesday June 23rd an individual investor buys some **Buoni del Tesoro Poliennali**, the Italian Treasury bonds, with face value of €25.000, coupon rate of 3,20%, and 3 remaining **half-yearly** coupons, the payment dates being April 1st and October 1st. Said bonds were issued at par; their quote is 99,83; the settlement date is Monday, June 28th. All coupons and any capital gain are taxed at 12,5% rate. The day count convention is **actual/actual**. Assume commissions and fees away and find the invoice price.

Solution.

Let time be measured in half-years. Since the latest coupon was paid on Thursday April 1st and the next will be paid on Friday October 1st, $29+31+28=88$ days have already gone by so that

$$t=\frac{88}{183}$$

Moreover, the coupon rate is a yearly nominal rate convertible half-yearly so that the gross and net coupons are worth $100*\frac{0{,}032}{2}=1{,}60$ and $1{,}60*0{,}875=1{,}40$. Reference must be made to the latter, since net coupons are paid to individual bondholders. Therefore, the accrued interest and dirty price are

$$1{,}40\frac{88}{183}=0{,}67 \text{ and } P_{dirty}=99{,}83+0{,}67=100{,}50$$

Since the dirty price of 1 unit of face value is $\frac{100{,}50}{100}$, the invoice price of 25.000 units of face value is $25.000\frac{100{,}50}{100}=25.125$ € .

REMARK.

For Treasury bonds (BTP) and Treasury certificates (CCT), the settlement date follows the issue or trade date by 2 business days; for corporate bonds traded in Italy, the settlement date follows the trade date by 2 business days. The day count convention **actual/actual** applies to all coupon bonds issued after 1/1/1999.

BTPs and CCTs are issued through regular electronic tenders conducted by the Bank of Italy. The tenders for BTPs and CCTs are **marginal**; in other words, all successful bids for a coupon bond are filled at the marginal price, i.e. the price of the last successful bid made by a financial intermediary. If BTPs or CCTs are subscribed at issue, no commissions are charged by financial intermediaries, who are paid by the Italian Treasury.

Exercise 31.

An individual investor buys some Treasury bonds with face value of €200.000, yearly coupons at a rate of 2,40%, and 5 years to maturity. The **gross yield to maturity** is 2% per year. Assume that each month has 30 days and find

a) today's clean price;

b) the clean price 3 months from now assuming that the gross yield to maturity doesn't change.

Said bonds were issued at par. All coupons and any capital gain are taxed at 12,5% rate, with commissions being 0,25% of face value. Find

c) the **net yield to maturity** per year;

d) the invoice price.

Solution.

Let time be measured in years. Each yearly **gross coupon** is worth $100*0{,}024 = 2{,}40$, whereas each **net coupon** is worth $100*0{,}024*0{,}875 = 2{,}10$.

a) As no interest has accrued yet, the unknown clean price satisfies the equation

$$P_{dirty} = P_{clean} = 2{,}40 a_{n|y} + 100*(1+y)^{-n}$$

where $n = 5$ is the number of remaining coupons and $y = 2\%$ is the **gross yield to maturity**. We have $P_{dirty} = P_{clean} = 101{,}885$.

b) The clean price 3 months from now is

$$P_{clean} = P_{dirty} - 2{,}40*0{,}25 = 101{,}885*1{,}02^{0,25} - 0{,}60 = 101{,}791$$

Remember that, if the gross yield to maturity *y* remains the same, the bond is an annuity with *n* remaining yearly non constant payments in arrears. The annuity value 1 year from now, equal to $P_{dirty}(1+y)$, is the sum of a future value, the next coupon, and a present value, the accompanying clean price.

c) The unknown **net yield to maturity** *y* satisfies the equation

$$P_{clean} = 101{,}885 + 0{,}250 = 102{,}135 = \left(2{,}10 a_{5|y} + 100(1+y)^{-5}\right)$$

which has no closed-form solution. Nonetheless, a **numerical** (**approximate**) solution can be obtained by using a spreadsheet package like Excel; we have $y = 1{,}652\%$ per year.

d) The invoice price of 200.000 units of face value is $200.000 \frac{102{,}135}{100} = 204.270$ € .

Exercise 32.

An individual investor buys some corporate bonds with face value of €100.000, yearly coupons at a rate of 5,76%, and 26 months to maturity. Said bonds were issued at par, today's quote being 101,34. All coupons and any capital gain are taxed at 20% rate, with commissions being 0,25% of face value. Assume that each month has 30 days and find

a) the invoice price;

b) the **gross** and **net** yield to maturity per year.

Solution.

Let time be measured in years. The yearly gross and net coupon are worth $100 * 0{,}0576 = 5{,}76$ and $5{,}76 * 0{,}8 = 4{,}608$. Since the latest coupon was paid 10 months ago, the accrued interest and dirty price are

$$4{,}608 \frac{10}{12} = 3{,}84 \quad \text{e} \quad P_{dirty} = 101{,}34 + 3{,}84 + 0{,}25 = 105{,}43$$

a) The invoice price of 100.000 units of face value is $100.000 \frac{105{,}43}{100} = 105.430$ € .

b) The unknown yields *y* satisfy the 2 equations

$$P_{dirty} = 101{,}34 + 5{,}76 \frac{10}{12} = 101{,}34 + 4{,}80 = 106{,}14 = \left(5{,}76 a_{3|y} + 100(1+y)^{-3}\right)(1+y)^{10/12}$$
$$P_{dirty} = 105{,}43 = \left(4{,}608 a_{3|y} + 100(1+y)^{-3}\right)(1+y)^{10/12}$$

which have no closed-form solutions. Nonetheless, 2 **numerical** (**approximate**) solutions can be obtained by using a spreadsheet package like Excel; the **gross** yield to maturity is $y = 5{,}081\%$ per year, whereas the **net** yield to maturity is $y = 3{,}823\%$ per year.

REMARK.

Consider the case of a **bondholder**, who is a **natural person** resident in Italy, is not an entrepreneur, and has opted for the widespread taxation system known as **administered saving**, introduced by the legislative decree No. 461 21/11/1997. His Treasury and corporate bonds are deposited with a bank or financial broker (SIM), not entrusted with their management.

According to the Italian tax law, a **substitute** tax is imposed by the financial intermediary on behalf of the tax office. Since **coupons**, **original issue discounts**, and **capital gains** are taxed at source, the financial income need not to be reported in the personal tax return. The tax rate for

Treasury securities, either domestic or appropriate foreign ones, and bonds issued by supranational entities is 12,5%, whereas the tax rate for corporate bonds is 26%.
According to the law decree No. 201 6/12/2011 a tax is levied on the market values of all bonds deposited with banks and financial brokers, the tax rate being 0,10% for 2012 and 0,15% for 2013. According to the law No. 147 27/12/2013 the tax rate is raised to 0,20% starting from 2014.

REMARK.
For simplicity's sake, suppose that the tax rate for corporate bonds is 20% rather than 26%. If the above-mentioned individual investor held the bond until expiry, he would incur a capital loss equal to 101,59-100=1,59, where 101,59 is the clean price inclusive of commissions. Each capital loss can be deducted from capital gains, also and not only on non-qualified shares, realised in the same business year and the 4 subsequent ones. If, *coeteris paribus*, the clean price had been $P_{clean} < 100$, inclusive of commissions, the individual investor would have benefited from a capital gain equal to $100 - P_{clean}$, either offset against appropriate capital losses or subject to a substitute tax equal to $(100 - P_{clean})0{,}20$.
More in general, let $\tilde{n}$ be the number of **yearly** coupons and $\tilde{P}_{clean}$ the clean price at **issue** of a corporate bond with face value of 100 percent and yearly coupon rate *c*; for $\tilde{P}_{clean} < 0$ there is an **original issue discount** equal to $100 - \tilde{P}_{clean}$, subject to a **withholding** tax at expiry equal to $(100 - \tilde{P}_{clean})0{,}20$. If the corporate bond is never traded, the entire tax is paid by the only holder; otherwise, the tax is paid by each holder commensurate with the respective holding period.
One peculiar but meaningful example will suffice to illustrate the general case; reconsider the case of a bondholder who purchases that corporate bond after issue and holds it until expiry. Let P_{clean} be the clean price inclusive of commissions, *n* the number of remaining coupons, and *t* the time passed since the latest coupon payment; the all-in clean price is then

$$P_{clean} - \max(100 - \tilde{P}_{clean}; 0)0{,}20\frac{\tilde{n} - (n - t)}{\tilde{n}}$$

where the second term is the tax on the original issue discount paid by the previous holder, with $\tilde{n} - (n - t)$ being the time passed since the bond issue. The dirty price is thus

$$P_{dirty} = P_{clean} - \max(100 - \tilde{P}_{clean}; 0)0{,}20\frac{\tilde{n} - (n - t)}{\tilde{n}} + 100c0{,}80t$$

As the corporate bond expires, the bondholder will

- receive the last net coupon, equal to $100c0{,}80$;
- receive the face value less the **withholding** tax on the whole original issue discount, i.e. an amount equal to $\max\left(100-\tilde{P}_{clean};0\right)0{,}20$;
- be likely to realise a capital gain or loss equal to $100-P_{clean}-\max\left(100-\tilde{P}_{clean};0\right)\frac{n-t}{\tilde{n}}$, to be treated in either of the above-mentioned manners. The third term is the portion of original issue discount accrued in the holding period $n-t$.

Therefore, when determining a possible capital gain or loss, the buying price and the portion of original issue discount accrued in between are taken away from the selling price; the buying (selling) price is a clean price plus (minus) commissions. If the same bond is purchased more times, the financial intermediary will consider the weighted average of all buying prices.

Exercise 33.

Consider a 4,20% coupon bond with face value of 100 per cent, half-yearly coupons, and 22 months to maturity. Suppose that its yield to maturity is 4,04% per year effective. Find

a) today's clean price;

b) the clean price 4 months from now assuming that its yield to maturity has not changed.

Solution.

Let time be measured in half years. The coupon rate is **convertible half-yearly** so that the **half-yearly coupon** is worth $100\frac{0{,}042}{2}=2{,}10$. Moreover, a 4,04% yield per year effective is equivalent to a 2% yield **per half year** $\left(1{,}02^2=1{,}0404\right)$.

a) Owing to consistency the dirty price satisfies the equation

$$P_{dirty}=\left(2{,}10a_{n|y2}+100*\left(1+y_2\right)^{-n}\right)\left(1+y_2\right)^t$$

where $n=4$ is the number of remaining coupons, $y_2=2\%$ is the yield **per half year**, and $t=\frac{60}{180}=\frac{1}{3}$ half years (i.e. 2 months) is the time passed since the latest coupon payment (or the bond issue). Therefore, the dirty and clean prices are

$$P_{dirty}=101{,}05 \quad \text{and} \quad P_{clean}=101{,}05-0{,}70=100{,}35$$

where $2{,}10t = \frac{2{,}10}{3} = 0{,}70$ is the accrued interest.

b) The clean price 4 months from now is

$$P_{clean} = 2{,}10 a_{3|2\%} + 100 * 1{,}02^{-3} = 100{,}35 * 1{,}02^{\frac{2}{3}} - 2{,}10 = 100{,}29$$

Remember that, if the yield to maturity y_2 remains the same, the bond is an annuity with *n* remaining half-yearly non constant payments in arrears. The annuity value 4 months from now, equal to $P_{dirty}(1+y_2)^{4/6}$, is the sum of a future value, the next coupon, and a present value, the accompanying clean price.

Exercise 34.

Consider a 3% coupon bond with face value of 100 percent, yearly coupons, and 10 years to maturity. Said bond is bought at today's quote of 91,89 and held until expiry. Find the accumulation 10 years from now and the corresponding **actual yearly yield** on the bond under the assumption that all coupons are reinvested at a rate of either 2%, or 3%, or 5%, or 6%.

Solution.

Let time be measured in years. The yearly coupon is worth $100 * 0{,}03 = 3$. Since the latest coupon has just been paid (or the bond has just been issued), the accrued interest is nought so that dirty and clean price are the same. The unknown **yearly yield to maturity** *y* satisfies the future value equation

$$91{,}89(1+y)^{10} = 3 s_{10|y} + 100$$

namely the appropriate present value equation with both sides multiplied by $(1+y)^{10}$. Using the appropriate built-in function of a spreadsheet package obtains $y = 4\%$.

The unknown **actual yearly yield** y_{ACT} satisfies the future value equation

$$91{,}89(1+y_{ACT})^{10} = 3 s_{10|i} + 100 = FV_{10}$$

where i is the yearly reinvestment rate and FV_{10} is the corresponding accumulation 10 years from now. We have

i	2%	3%	4%	5%	6%
FV_{10}	132,85	134,39	136,02	137,73	139,54
y_{ACT}	3,755%	3,875%	4%	4,130%	4,266%

REMARK.

If the bond is sold before its expiry, e.g. 4 years from now at a clean price P_{clean}, the actual yearly yield y_{ACT} meets the future value equation

$$91{,}89(1+y_{ACT})^4 = 3s_{4|i} + P_{clean} = FV_4$$

In that case, the bondholder faces a **reinvestment rate risk** as well as a **price risk**.

Exercise 35.

Consider a 9% Treasury bond with face value of 100 percent, yearly coupons, and 10 years to maturity. Said bond is bought at today's quote of 147,89 and held for 2 years; coupons are not reinvested. Find the accumulation 2 years from now and the corresponding **actual yearly yield** on the bond under the assumption that the bond is resold at a quote of either 139,51, or 147,89, or 154,57.

Solution.

Let time be measured in years. The yearly coupon is worth $100*0{,}09=9$. Since the latest coupon has just been paid (or the bond has just been issued), the accrued interest is nought so that dirty and clean price are the same. The unknown **yearly yield to maturity** y satisfies the present value equation

$$147{,}89 = 9a_{10|y} + 100(1+y)^{-10}$$

Using the appropriate built-in function of a spreadsheet package obtains $y=3{,}3\%$. The implicit quote 2 years from now is

$$139{,}51 = 9a_{8|3{,}3\%} + 100*1{,}033^{-8}$$

The unknown **actual yearly yield** y_{ACT} satisfies the future value equation

$$147{,}89(1+y_{ACT})^2 = 9*2+P_{clean} = FV_2$$

where P_{clean} is the selling price and FV_2 is the corresponding accumulation 2 years from now. We have

P_{clean}	139,51	147,89	154,57
FV_2	157,51	165,89	172,57
y_{ACT}	3,201%	5,911%	8,022%

REMARK.

The bondholder faces a **reinvestment rate risk** as well as a **price risk**; the former is disregarded.

REMARK.

The latter author is grateful to Ugo Gardella (1927-2018), a chemical engineer, who told him about trading opportunities like the one considered in Exercise **35**.

4.2. Financial duration

The Macaulay duration of a bond approximately measures the sensitivity of the dirty price P_{dirty} to a sudden and mild change in the **yearly** yield to maturity *y* (Luenberger, 1998, p. 58). Let time *t* be measured in years and time 0 be the appraisal date. First consider the general case of an annuity, namely a sequence of regular payments due at the times $t_1 < t_2 < \cdots < t_n$; the **financial duration** *D* of the annuity is

$$D = \sum_{k=1}^{n} t_k \frac{PV_k}{PV} = \frac{\sum_{k=1}^{n} t_k PV_k}{\sum_{k=1}^{n} PV_k}$$

where PV_k is the present value of the payment due at time t_k and *PV* is the present value of the annuity. Although the sensitivity analysis and the 2 propositions below make reference to bonds, *mutatis mutandis*, they carry over to annuities as well.

REMARK.

Consider the peculiar case of an **immediate annuity** with *n* constant **periodic** payments in arrears *R*, made *m* times per year; we have $t_1 = \frac{1}{m} < t_2 = \frac{2}{m} < \cdots < t_n = \frac{n}{m}$ and

$$D = \sum_{k=1}^{n} t_k \frac{PV_k}{PV} = \frac{\sum_{k=1}^{n} \frac{k}{m} R(1+i_m)^{-k}}{R a_{n|i_m}}$$

where i_m is the periodic rate of interest charged on the annuity. It can be proved that the general formula becomes

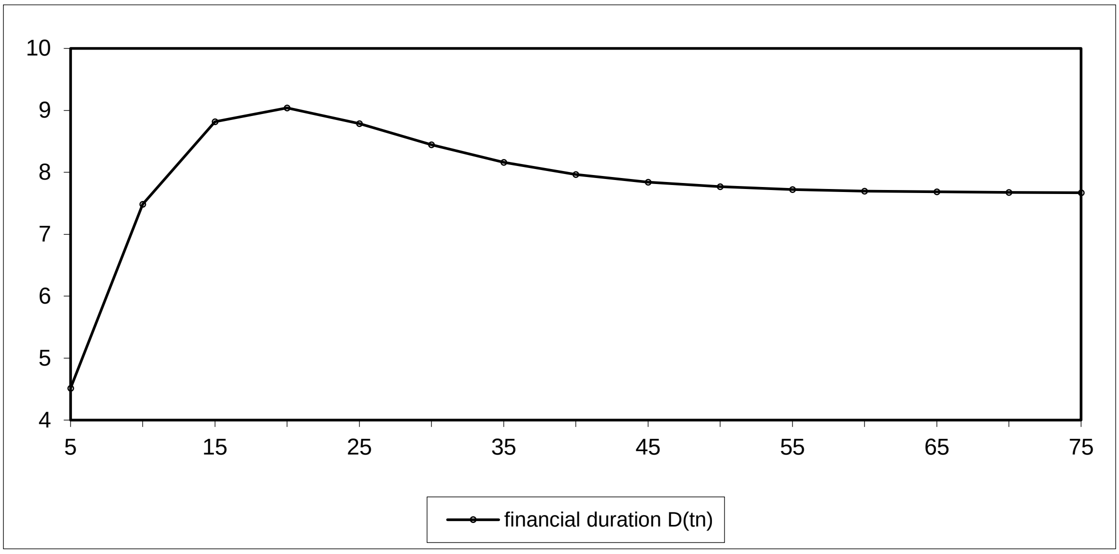
$$D = \left(\frac{1+i_m}{i_m} - \frac{n}{(1+i_m)^n - 1} \right) \frac{1}{m} \quad \text{years}$$

When it comes to a coupon bond, the terms of the above general formula become

$$PV = P_{dirty} \quad \text{and} \quad PV_k = \begin{cases} 100c(1+y)^{-t_k} & \text{for} \quad t_k < t_n \\ 100(1+c)(1+y)^{-t_n} & \text{for} \quad t_k = t_n \end{cases}$$

as each payment is a (yearly, half-yearly, or quarterly) coupon, also including the face value at the expiry t_n. As the Macaulay duration D is a weighted average of all payment dates, the weights being the ratios $\frac{PV_k}{P_{dirty}}$, we have $t_1 \le D \le t_n$ with $D = t_1$ for a pure discount bond and $D < t_n$ for a coupon bond. It can be proved that, *coeteris paribus*,

- the larger the (periodic) coupon rate *c*, the smaller the duration *D*;
- the larger the yield to maturity *y*, the smaller the duration *D*;
- for $t_n \to +\infty$ the financial duration *D* tends to a finite limit. Moreover, for $y \le c$, the larger t_n, the larger *D*, whereas for $y > c$ and a less distant (more distant) t_n, the larger t_n, the larger (smaller) *D*. A case in point is portrayed in the drawing below, where $c = 4\%$ per year, $y = 15\%$ per year, and t_n ranges from 5 to 75 years.



Let $P_{dirty} = f(y)$ represent the dirty price as a function of the yield to maturity. If a Taylor's expansion of $f(y)$ truncated at 1st order is considered

$$\Delta P_{dirty} = f'(y)\Delta y + \frac{f''(\tilde{y})}{2}\Delta y^2 = -DP_{dirty}\frac{\Delta y}{1+y} + \frac{f''(\tilde{y})}{2}\Delta y^2$$

where $\tilde{y}$ is an internal point of the interval $[y; y+\Delta y]$, it is readily ascertained that the sensitivity of P_{dirty} to changes in *y* can be approximately represented by the **linear** term

$$\frac{\Delta P_{dirty}}{P_{dirty}} \cong -\frac{D}{1+y}\Delta y = -\tilde{D}\Delta y$$

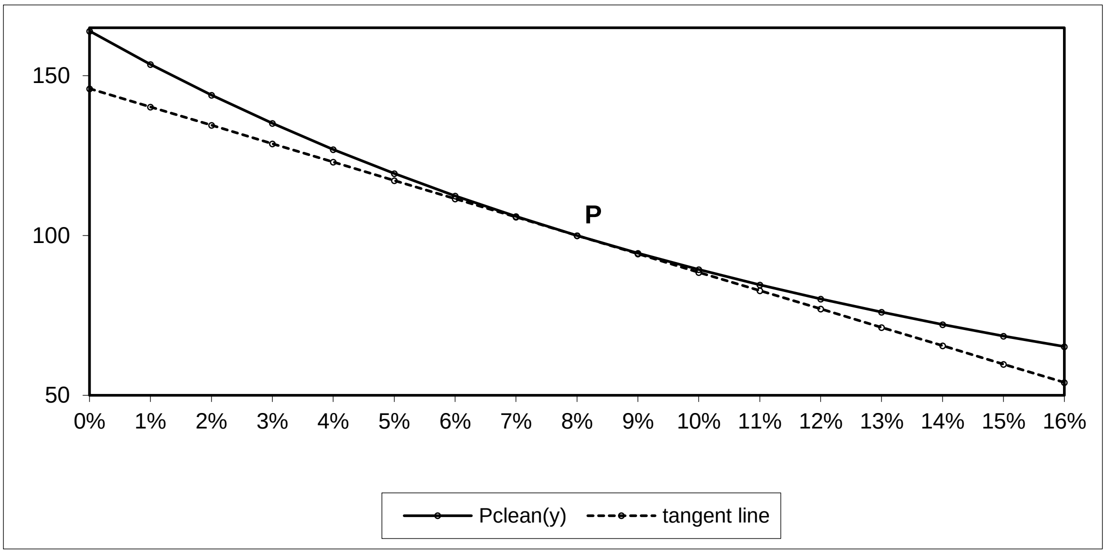
with the approximation error being equal to the Taylor's expansion remainder. Roughly speaking, a sudden and mild **rise** (**drop**) in *y* would bring about a **capital loss** (**gain**) that is proportional to the **modified financial duration** $\tilde{D}$.

Example 19.

Consider an 8% coupon bond with yearly coupons and 8 years to maturity. Suppose that the bond is bought **at par** at issue. Therefore, the yield to maturity at issue is 8%, whereas the financial duration is $D = 6{,}206$. The $P_{dirty} = f(y)$ curve is plotted in the figure below, where point *P* marks the initial conditions. The straight line

$$P_{dirty} - 100 = -\frac{6{,}206 * 100}{1{,}08}(y - 0{,}08)$$

is tangent to the $P_{dirty} = f(y)$ curve at point *P*. To anticipate the impact on P_{dirty} of a sudden and mild change in *y*, use can be made of such tangent straight line, i.e. a **linear** approximation. For instance, if *y* rose to 8,50%, there would be an approximate capital loss of 2,873 with P_{dirty} dropping to 97,127 or so.



REMARK.

It is readily ascertained that the actual dirty price after the change in yield to maturity is equal to 97,180. Therefore, the approximation error is definitely small, being equal to $97{,}127 - 97{,}180 = -0{,}053$, which explains why the above-mentioned calculation is often made by market practioners to assess how a +1% (-1%) change in a bond yield to maturity would impact on the bond dirty price. It is also readily ascertained that *coeteris paribus* (all other things being equal), the more distant the bond maturity *n*, the less accurate the approximation is.

Let's recapitulate: the price of a bond can change owing to either the **passage of time** or a **change** in the **yield to maturity**; the statement applies to the financial duration as well.

Proposition.

As long as *y* does not change and no payment is made, *D* decreases linearly as time goes by.

PROOF.

Let *t* be the appraisal date with $0 \le t < t_1$. The financial duration at time *t* is

$$\frac{\sum_{k=1}^{n}(t_k - t)PV_k(1+y)^t}{\sum_{k=1}^{n}PV_k(1+y)^t} = \frac{\sum_{k=1}^{n}(t_k - t)PV_k}{\sum_{k=1}^{n}PV_k} = \frac{\sum_{k=1}^{n}t_k PV_k}{\sum_{k=1}^{n}PV_k} - t = D - t$$

Suppose that different amounts of money are invested in different bonds that have the **same** yearly yield to maturity *y*. Let t_n be their furthest maturity. The portfolio value *PV* obeys an equation like

$$PV = PV_A + PV_B + PV_C = C_A \frac{P_{dirtyA}}{100} + C_B \frac{P_{dirtyB}}{100} + \frac{C_C}{100} P_{dirtyC}$$

where C_A is the face value of bonds *A* in the bond portfolio.

Proposition.

The portfolio financial duration is

$$D = w_A D_A + w_B D_B + w_C D_C$$

where the weights w_A, w_B, w_C are the proportions invested in the 3 bonds.

PROOF.

We have

$$D = \sum_{k=1}^{n} \frac{t_k \left(PV_{A_k} + PV_{B_k} + PV_{C_k}\right)}{PV} = \sum_{k=1}^{n} \frac{t_k \left(PV_{A_k} + PV_{B_k} + PV_{C_k}\right)}{PV_A + PV_B + PV_C}$$

and hence

$$D = \sum_{k=1}^{n} \frac{t_k\, PV_{A_k}}{PV_A} \frac{PV_A}{PV_A + PV_B + PV_C} + \sum_{k=1}^{n} \frac{t_k\, PV_{B_k}}{PV_B} \frac{PV_B}{PV_A + PV_B + PV_C} + \sum_{k=1}^{n} \frac{t_k\, PV_{C_k}}{PV_C} \frac{PV_C}{PV_A + PV_B + PV_C} =$$
$$= w_A D_A + w_B D_B + w_C D_C$$

If the yield to maturity *y* changes by a small amount, the resulting change in the bond portfolio value will be approximated by the above-mentioned linear equation. The notion of portfolio financial duration is used in practice: if bond managers expect all interest rates to drop (rise), they will lengthen (shorten) the portfolio duration appropriately. Indeed, if a bond portfolio has a fairly large (small) duration, its value is (un)favourably affected by such a change.

Now consider a portfolio that includes both **assets** and **liabilities**, with all assets (liabilities) having the same yield to maturity y_A (y_L); let PV_A (PV_L) be the present value of all assets (liabilities) and D_A (D_L) their financial duration. If D_A and D_L are suitably matched, **immunisation** is achieved against **interest rate risk**. In other words, if all yields to maturity undergo the same small change $\Delta y = \Delta y_A = \Delta y_L$, we approximately have $\Delta PV_A \cong \Delta PV_L$ so that no equity capital is lost.

For instance, a **life insurance company** can structure an asset portfolio to meet some liabilities, i.e. some promised payments to policy-holders. Furthermore, a **bank** can take in new deposits and issue new securities to fund new bilateral loans; needless to say, the former are liabilities, whereas the latter are assets. Since deposits can be withdrawn at short notice, whereas many loans are long term, the bank faces a so called **maturity transformation** problem.

Convexity

The convexity of a bond measures the degree of curvature of the price-yield to maturity function. It can be used together with the financial duration to obtain a better approximation of the change in the dirty price P_{dirty} due to a sudden change in the **yearly** yield to maturity *y*. Let time *t* be measured in years, $t_1 < t_2 < \cdots < t_n$ be payment dates and time 0 be the appraisal date. The **convexity** $\tilde{C}$ of a bond is

$$\tilde{C} = \frac{1}{(1+y)^2} \frac{\sum_{k=1}^{n} \left(t_k^2 + t_k\right) PV_k}{P_{dirty}}$$

where PV_k is the present value of the payment due at time t_k, i.e.

$$PV_k = \begin{cases} 100c(1+y)^{-t_k} & \text{for} \quad t_k < t_n \\ 100(1+c)(1+y)^{-t_n} & \text{for} \quad t_k = t_n \end{cases}$$

as each payment is a (yearly, half-yearly, or quarterly) coupon, also including the face value at the expiry t_n. It can be proved that, *coeteris paribus*,

- the larger the (periodic) coupon rate *c*, the smaller the convexity $\tilde{C}$;

- the larger the yield to maturity y, the smaller the convexity $\tilde{C}$.

Moreover, zero coupon bonds have the least convexity $\tilde{C}$ among all bonds with equal yield to maturity y and financial duration D. Actually, coupon bonds have more dispersed payments.

Let $P_{dirty} = f(y)$ represent the dirty price as a function of the yield to maturity. If a Taylor's expansion of $f(y)$ truncated at 2nd order is considered

$$\Delta P_{dirty} = f'(y)\Delta y + \frac{f''(y)}{2}\Delta y^2 + \frac{f'''(\tilde{y})}{6}\Delta y^3 = -DP_{dirty}\frac{\Delta y}{1+y} + \frac{\tilde{C}P_{dirty}}{2}\Delta y^2 + \frac{f'''(\tilde{y})}{6}\Delta y^3$$

where $\tilde{y}$ is an internal point of the interval $[y; y+\Delta y]$, it is readily ascertained that the sensitivity of P_{dirty} to changes in y can be approximately represented by the **linear** and **quadratic** terms

$$\frac{\Delta P_{dirty}}{P_{dirty}} \cong -\frac{D}{1+y}\Delta y + \frac{\tilde{C}}{2}\Delta y^2 = -\tilde{D}\Delta y + \frac{\tilde{C}}{2}\Delta y^2$$

with the approximation error being equal to the Taylor's expansion remainder.

REMARK.

Consider several bonds with the same yield to maturity y and financial duration D. It is readily realised that the larger the convexity $\tilde{C}$, the larger the change in the dirty price P_{dirty} due to a change in the yield to maturity y. If reference were made to a **non-flat** term structure of interest rates and several bonds with the same financial duration were considered, the previous proposition could be restated as follows: the larger the convexity, the larger the change in the dirty price due to a **parallel** shift in the term structure.

Example 20.

Consider an 8% coupon bond with yearly coupons and 8 years to maturity, as in Example **19**. Suppose that the bond is bought **at par** at issue. Therefore, the yield to maturity at issue is 8%, whereas the financial duration and convexity are $D = 6{,}206$ and $\tilde{C} = 43{,}616$. The parabola

$$P_{dirty} - 100 = -\frac{6{,}206*100}{1{,}08}(y - 0{,}08) + \frac{43{,}616*100}{2}(y - 0{,}08)^2$$

is tangent to the $P_{dirty} = f(y)$ curve at point $P = (0{,}08;100)$. To anticipate the impact on P_{dirty} of a sudden change in *y*, use can be made of such tangent parabola, i.e. a **quadratic** approximation. For instance, if *y* rose to 8,50%, there would be an approximate capital loss of 2,819 with P_{dirty} dropping to 97,181 or so. Recall that the actual dirty price after the change in yield to maturity is 97,180.

In the following table, reference is still made to an 8% coupon bond with yearly coupons and 8 years to maturity. Different changes Δy of the yearly yield to maturity *y* are considered, with the actual bond price P_{dirty} being compared with its linear and quadratic approximation. The former is based on the financial duration, whereas the latter includes the convexity as well. For small changes Δy of the yearly yield to maturity *y* the linear approximation is fairly accurate, whereas for larger changes Δy the quadratic approximation is to be preferred.

Δy	P_{dirty}	Linear approx.	Error	Quadratic approx.	Error
-5%	135,098	128,731	-6,367	134,183	-0,915
-3%	119,390	117,239	-2,151	119,202	-0,188
-1%	105,971	105,746	-0,225	105,964	-0,007
-0,5%	102,929	102,873	-0,056	102,928	-0,001
0,5%	97,180	97,127	-0,053	97,181	0,001
1%	94,465	94,254	-0,211	94,472	0,007
3%	84,562	82,761	-1,801	84,724	0,162
5%	76,006	71,269	-4,737	76,721	0,715

Suppose that different amounts of money are invested in different bonds that have the **same yearly** yield to maturity *y*. Let t_n be their furthest maturity. The portfolio value *PV* obeys an equation like

$$PV = PV_A + PV_B + PV_C = C_A \frac{P_{dirtyA}}{100} + C_B \frac{P_{dirtyB}}{100} + \frac{C_C}{100} P_{dirtyC}$$

where C_A is the face value of bonds *A* in the bond portfolio.

Proposition.

The portfolio convexity is

$$\tilde{C} = w_A \tilde{C}_A + w_B \tilde{C}_B + w_C \tilde{C}_C$$

where the weights w_A, w_B, w_C are the proportions invested in the 3 bonds.

PROOF.

We have

$$\tilde{C} = \sum_{k=1}^{n} \frac{\left(t_k^2 + t_k\right)\left(PV_{A_k} + PV_{B_k} + PV_{C_k}\right)}{(1+y)^2 PV}$$

and hence

$$\tilde{C} = \sum_{k=1}^{n} \frac{\left(t_k^2 + t_k\right) PV_{A_k}}{(1+y)^2 PV_A} \frac{PV_A}{PV} + \sum_{k=1}^{n} \frac{\left(t_k^2 + t_k\right) PV_{B_k}}{(1+y)^2 PV_B} \frac{PV_B}{PV} + \sum_{k=1}^{n} \frac{\left(t_k^2 + t_k\right) PV_{C_k}}{(1+y)^2 PV_C} \frac{PV_C}{PV} =$$
$$= w_A \tilde{C}_A + w_B \tilde{C}_B + w_C \tilde{C}_C$$

If the yield to maturity y changes by a small amount, the resulting change in the bond portfolio value will be approximated by the above-mentioned quadratic equation. The notion of convexity is used in asset management as well as in asset and liability management.

Exercise 36.

Consider a 5% coupon bond with face value of 100 percent, yearly coupons, and 27 months to maturity. Suppose that each month has 30 days and the yield to maturity is 5% per year effective. Find

a) financial duration and convexity.

Suppose that the yield to maturity suddenly changes from 5% to 6%. Find

b) a **linear approximation** of the change in the dirty price;

c) a **quadratic approximation** of the change in the dirty price.

Solution.

Let time be measured in years.

a) Owing to consistency the dirty price satisfies the equation

$$P_{dirty} = \left(5a_{n|y} + 100 * (1+y)^{-n}\right)(1+y)^t$$

where $n = 3$ is the number of remaining coupons, $y = 5\%$ is the yield **per year**, and $t = 0{,}75$ years (i.e. 9 months) is the time passed since the latest coupon payment (or the bond issue). Therefore, the dirty price is $P_{dirty} = 103{,}727$. Moreover, financial duration and convexity are

$$D = \frac{0,25*5*1,05^{-0,25} + 1,25*5*1,05^{-1,25} + 2,25*105*1,05^{-2,25}}{P_{dirty}} = 2,109 \text{ years} =$$
$$= 2 + 0,109*360 = 2 \text{ years and } 39 \text{ days}$$

$$\tilde{C} = \frac{\left(0,25^2 + 0,25\right)*5*1,05^{-0,25} + \left(1,25^2 + 1,25\right)*5*1,05^{-1,25} + \left(2,25^2 + 2,25\right)*105*1,05^{-2,25}}{1,05^2 P_{dirty}} = 6,145$$

b) The dirty price must drop, since all remaining payments are discounted at a larger yearly rate $y + \Delta y = 6\%$. Its change can be estimated as

$$\Delta P_{dirty} \cong -\frac{DP_{dirty}\Delta y}{1+y} = -\frac{2,109*103,727*0,01}{1,05} = -2,083$$

so that its new **approximate** value is $P_{dirty} + \Delta P_{dirty} \cong 103,727 - 2,083 = 101,644$.

c) We have

$$\Delta P_{dirty} \cong -\frac{DP_{dirty}\Delta y}{1+y} + \frac{\tilde{C}P_{dirty}\Delta y^2}{2} = -\frac{2,109*103,727*0,01}{1,05} + \frac{6,145*103,727*0,01^2}{2} = -2,052$$

and hence the more accurate approximation $P_{dirty} + \Delta P_{dirty} \cong 103,727 - 2,052 = 101,675$.

It is readily ascertained that the actual dirty price after the change in yield to maturity is equal to 101,675.

REMARK.

Recall that as long as y does not change over time and no payment is made, the financial duration decreases linearly as time goes by, which proves helpful when checking for computation errors. In the case under examination this implies that

$$D \text{ 27 months before maturity} = D \text{ 3 years before maturity} - 9 \text{ months} = 2,859 - 0,75 = 2,109$$

Exercise 37.

Consider a 6% coupon bond with face value of 100 percent, half-yearly coupons, and 9 months to maturity. Suppose that each month has 30 days and the yield to maturity is 6,09% per year effective. Find

a) the financial duration;

b) a **linear approximation** of the change in the dirty price under the assumption that the yield to maturity suddenly changes from 6,09% to 5,59%.

Hint: use the half year (the year) as unit of time when determining the dirty price (the financial duration and the approximate change in the dirty price).

Solution.

a) Let time be measured in **half years**. The coupon rate is **convertible half-yearly** so that the **half-yearly coupon** is worth $100\frac{0{,}06}{2}=3$. Moreover, a 6,09% yield per year effective is equivalent to a 3% yield **per half year** $\left(1{,}03^2=1{,}0609\right)$. Owing to consistency the dirty price satisfies the equation

$$P_{dirty}=\left(3a_{n|y2}+100*\left(1+y_2\right)^{-n}\right)\left(1+y_2\right)^t$$

where $n=2$ is the number of remaining coupons, $y_2=3\%$ is the yield **per half year**, and $t=0{,}5$ half years (i.e. 3 months) is the time passed since the latest coupon payment (or the bond issue). Therefore, the dirty price is

$$P_{dirty}=101{,}49$$

Let time be measured in **years.** The financial duration takes the value

$$D=\frac{0{,}25*3*1{,}0609^{-0{,}25}+0{,}75*103*1{,}0609^{-0{,}75}}{P_{dirty}}=0{,}735\ \text{years}=0{,}735*360=265\ \text{days}$$

b) The dirty price must rise, since all remaining payments are discounted at a smaller yearly rate $y+\Delta y=5{,}59\%$. Its change can be estimated as

$$\Delta P_{dirty}\cong-\frac{DP_{dirty}\Delta y}{1+y}=-\frac{0{,}735*101{,}49*(-0{,}005)}{1{,}0609}=+0{,}35$$

so that its new **approximate** value is $P_{dirty}+\Delta P_{dirty}\cong101{,}49+0{,}35=101{,}84$.

Exercise 38.

A portfolio includes the following bonds

	coupon rate	years to maturity	dirty price	yield to maturity	financial duration	convexity
bond A	5%	6	105,24	4%	5,349	33,249
bond B	4%	3,5	101,98	4%	3,275	13,388

Coupons are paid yearly; bond A (B) has a face value of €40.000 (€60.000). Find

a) the portfolio value, the portfolio financial duration, and the portfolio convexity.

Suppose that the yearly yield to maturity suddenly changes from 4% to 3,5%. Find

b) a **linear approximation** of the change in the portfolio value;

c) a **quadratic approximation** of the change in the portfolio value.

Solution.

Let time be measured in years. Recall that immunisation theory in its simplest version assumes that all bonds have the same yield to maturity *y*.

a) The portfolio value is

$$PV = 40.000\frac{105{,}24}{100} + 60.000\frac{101{,}98}{100} = 42.096 + 61.188 = 103.284 \text{ €}$$

whereas the portfolio financial duration is

$$D = \frac{42.096}{103.284}5{,}349 + \frac{61.188}{103.284}3{,}275 = 4{,}120 \text{ years}$$

and the convexity is

$$\tilde{C} = \frac{42.096}{103.284}33{,}249 + \frac{61.188}{103.284}13{,}388 = 21{,}483$$

b) The portfolio value must rise, since all remaining payments are discounted at a smaller yield to maturity $y + \Delta y = 3{,}5\%$. Its change can be estimated as

$$\Delta PV \cong -\frac{DPV\Delta y}{1+y} = -\frac{4{,}120 * 103.284 * (-0{,}005)}{1{,}04} = 2.045{,}82$$

so that its new **approximate** value is

$$PV + \Delta PV \cong 103.284 + 2.045{,}82 = 105.329{,}82 \text{ €}$$

c) We have

$$\Delta PV \cong -\frac{DPV\Delta y}{1+y}+\frac{\tilde{C}PV\Delta y^2}{2}=-\frac{4{,}120*103.284*(-0{,}005)}{1{,}04}+\frac{21{,}483*103.284*0{,}005^2}{2}=2.073{,}55$$

and hence the more accurate approximation $PV+\Delta PV \cong 103.284+2.073{,}55=105.357{,}55$ €.

Exercise 39.

A liability of €100.000 due 4 years from now is to be met by managing an **imaginary** asset portfolio that includes the following bonds

	yearly coupon rate	years to maturity	clean price	yearly yield to maturity	financial duration
bond A	6%	6	105,08	5%	5,234
bond B	5%	4	100,00	5%	3,723

Coupons are paid yearly and any amount of each bond can be purchased.

a) Find the asset allocation that achieves immunisation against interest rate risk, i.e. against a (small) change in yield to maturity.

b) Suppose that the yearly yield to maturity does not change. Find the portfolio value 1 year from now.

c) Suppose that the yearly yield to maturity drops from 5% to 4% immediately after forming the asset portfolio. Check that immunisation is effective and determine how the portfolio has to be rebalanced.

Solution.

Let time be measured in years, *PV* be present value, *y* be yield to maturity, *D* be financial duration. Recall that immunisation theory in its simplest version assumes that there is a **single** yearly yield to maturity for all assets and liabilities.

a) The asset portfolio must have **present value** and **financial duration** equal to those of the liability. In this case a (small) change Δy in yield would approximately bring about the same change in the present values of assets and liability

$$\Delta PV \cong -D\,PV\frac{\Delta y}{1+y}$$

If this happens (or a coupon is paid), the asset portfolio must be rebalanced, which does not nearly require any expense (transaction costs are not considered). Therefore, the asset portfolio satisfies the pair of linear equations

$$PV_A + PV_B = PV = 100.000*1{,}05^{-4} = 82.270{,}25$$
$$\frac{PV_A}{PV}5{,}234 + \frac{PV_B}{PV}3{,}723 = 4$$

the solution to which is $PV_A = 15.081{,}97$ €; $PV_B = 67.188{,}28$ €. In other words, €15.081,97 (€67.188,28) must be invested in bond A (B).

b) If the yearly yield to maturity *y* does not change, the liability present value *PV* after 1 year is worth

$$PV = 82.270{,}25*1{,}05 = 100.000*1{,}05^{-3} = 86.383{,}76$$

Owing to **consistency**, this is the asset portfolio value as well, i.e. the sum of the next coupons and the present value of all subsequent payments.

c) Since immunisation is based on a linear approximation, there is no perfect hedge. We have

$$15.081{,}97\frac{110{,}48}{105{,}08} + 67.188{,}28\frac{103{,}63}{100{,}00} - 100.000*1{,}04^{-4} =$$
$$= 15.857{,}02 + 69.627{,}21 - 85.480{,}42 = 3{,}81 \text{ €}$$

where 110,48 (103,63) is the clean price of bond A (B) at a 4% yield; both assets and liability have thus appreciated. In this circumstance immunisation proves effective. However, the financial duration of bond A (B) at a 4% yield is 5,256 (3,729); immunisation is lost but can be restored by rebalancing the asset portfolio so as to satisfy the pair of linear equations

$$PV_A + PV_B = PV = 100.000*1{,}04^{-4} = 85.480{,}42$$
$$\frac{PV_A}{PV}5{,}256 + \frac{PV_B}{PV}3{,}729 = 4$$

the solution to which is $PV_A = 15.170{,}40$ €; $PV_B = 70.310{,}02$ €. Since €15.170,40 (70.310,02) must be invested in bond A (B), 2 transactions are required: sell €686,62 of bond A and buy €682,81 of bond B.

REMARK.

Therefore, if the asset portfolio is rebalanced whenever appropriate, assets and liability have always the same present value whether the yearly yield to maturity remains the same or not. This is also true when the liability expires and the assets are sold so as to meet the ensuing obligation.

REMARK.

Suppose that a single liability is matched by a portfolio of assets with different maturities such that assets and liability have the same present value $PV_A(0;y) = PV_L(0;y)$ and financial duration $D_A = D_L$. It can be proved that any **finite** change Δy in the yield to maturity y is such that $\frac{PV_A(0^+;y+\Delta y)}{PV_L(0^+;y+\Delta y)} > 1$. This finding is due to Frank Mitchell Redington, a British actuary who worked all his life with Prudential life insurance (McCutcheon-Scott , 1986, p. 243).

This implication carries over to the cases of

- a **parallel** shift in a **non-flat** term structure of interest rates;
- several liabilities with different maturities, provided that assets are altogether more dispersed in time than liabilities, with dispersions taking the form of suitable mean absolute deviations.

However, the assumption of **parallel** shifts in a term structure of interest rates is neither realistic nor theoretically sound, even if such frictions as taxes, constraints on short positions, commissions, fees, bid-ask spreads are assumed away. One can benefit from an arbitrage opportunity indeed, e.g, by buying 2 zero coupon bonds and selling short a zero coupon bond by the same amount, provided that the maturity of the liability is equal to the duration of the asset portfolio.

Exercise 40.

Consider an **imaginary** bank portfolio that includes business loans to be funded with a **liability** and some **equity capital**. As for the **assets**, the amount lent is equal to €5.000.000, whereas the nominal rate of interest charged is 8% per year convertible half-yearly; the repayment schedule includes 20 half-yearly instalments, each worth €367.908,75, their financial duration being 4,6046 years. The **liability** takes the form of a 4% bond, to be issued at par, with 12 half-yearly coupons, the financial duration being 5,3934 years.

a) Find the present value of the **liability** that achieves immunisation against interest rate risk, i.e. against a small and equal change in both yields to maturity.

b) Suppose that neither yield to maturity changes; find the portfolio value 3 months from now.

Solution.

Let time be measured in years, *E* be equity capital, *PV* be present value, *y* be yield to maturity, *D* be financial duration. Recall that immunisation theory in a more advanced version assumes that all assets (liabilities) have the **same** yearly yield to maturity y_A (y_L).

a) We have $y_A = 1{,}04^2 - 1 = 0{,}08160$ and $y_L = 1{,}02^2 - 1 = 0{,}04040$. Let $E = PV_A - PV_L$; a small change $\Delta y = \Delta y_A = \Delta y_L$ would approximately bring about the following change in *E*

$$\Delta E = \Delta PV_A - \Delta PV_L \cong \left(-\frac{D_A}{1+y_A} PV_A + \frac{D_L}{1+y_L} PV_L \right) \Delta y$$

If this happens (or 2 payments are made), rebalancing is needed. Therefore, in order to achieve immunisation, i.e. $\Delta E \cong 0$, the following equation must be met

$$\frac{D_A}{1+y_A} PV_A = \frac{D_L}{1+y_L} PV_L \quad \text{namely} \quad \frac{4{,}6046}{1{,}08160} 5.000.000 = \frac{5{,}3934}{1{,}04040} PV_L$$

so that $PV_L = 4.106.150$ € and hence $E = PV_A - PV_L = 893.850$ €. Since the coupon bond is issued at par, $PV_L = 4.106.150$ € is its face value as well.

b) If neither yield to maturity changes and no payment is made, the present values of all assets and all liabilities after 3 months are

$$PV_A = 5.000.000 * 1{,}08160^{0{,}25} = 5.099.019{,}51 \text{ €}$$
$$PV_L = 4.106.150{,}00 * 1{,}04060^{0{,}25} = 4.147.008{,}22 \text{ €}$$

so that $E = PV_A - PV_L = 952.011{,}29$€, their financial durations being

$$D_A = 4{,}6046 - 0{,}2500 = 4{,}3546 \quad \text{e} \quad D_L = 5{,}3934 - 0{,}2500 = 5{,}1434$$

Therefore, the yield spread brings about an equity capital gain equal to €58.161,29; however, immunisation is lost as time goes by so that portfolio rebalancing is sooner or later needed.

REMARK.

In order to finalise the previous exercise, use has been made of a Tailor's expansion of the function $E = f(y_A; y_L)$ truncated at 2nd order. The sign of its **quadratic** term depends on PV_A and y_A, PV_L and y_L, as well as on the **convexities** of assets and liability.

4.3. Assessment of credit risk by credit-rating agencies

International **credit-rating agencies** group companies into **credit-risk classes** according to their creditworthiness, assessed in the middle of a business cycle so as to carry out updates as rarely as possible. The best known credit-rating agencies are Fitch Ratings, Moody's Investors Service, and Standard & Poor's. Companies do pay credit-rating agencies for bringing down the informational asymmetry between potential lenders and their top managers so as to benefit from lower coupon rates when issuing their own corporate bonds. If more credit-rating agencies are hired by a company, their ratings may be split. Potential lenders rather than companies used to pay credit-rating agencies until the 1970s, when photocopiers came into use. A usual, qualitative and simplified representation of credit-risk classes is reported in Table 3 below, which is made up of 2 halves, one of **investment grade** bonds and the other of **speculative grade** bonds. Cautious bondholders are concerned with the first half of the table only.

Major class	**Bond (issuer) description**
AAA	Extremely strong capacity to repay principal and interest
AA	Very strong capacity to repay principal and interest
A	Strong capacity to repay principal and interest, but somewhat more susceptible to adverse macro and microeconomic conditions
BBB	Adequate capacity to repay principal and interest, but more subject to adverse conditions.
BB; B **CCC; CC**	Speculative-investment bonds
C	No interest is being paid
D	Bond in default

Table 3 – Medium-term rating scale according to S&P's

A **medium-term** rating of a bond issuer comes along with a **short-term** rating of all short-term financial obligations, with the time horizon of the former (latter) being 3-5 years (13 months). A lower rating implies a worser quality, called **creditworthiness**, which is matched by a higher default risk and hence by a higher yield to maturity as a compensation.

REMARK.

Standard & Poor's was established in 1860 to provide information on US railway companies. It was acquired in 1966 by McGraw-Hill, a major publishing company. Its network of offices is based in 23 different countries.

Some empirical estimates of **default** and **recovery rates** for each credit-risk class are reported in the tables below. Default and recovery rates are affected by the business cycle, with the former (latter) rising (dropping) in a recession as well as before (after) the 7/90-3/91 and 4/01-12/01 recessions. Moreover, default rates are industry dependent, with technical advance being a key driving factor; utilities, banks, and insurance companies display the lowest yearly averages and standard deviations over the medium-long term.

S&P's rating	**Years since issue**						
	1	**2**	**3**	**4**	**5**	**7**	**10**
AAA	0,00%	0,00%	0,00%	0,00%	0,00%	0,13%	0,13%
AA	0,00%	0,00%	1,81%	2,20%	2,33%	2,33%	2,46%
A	0,00%	0,31%	0,71%	0,71%	0,71%	0,89%	0,93%
BBB	0,04%	0,29%	0,46%	0,46%	0,91%	1,07%	2,12%
BB	0,00%	0,62%	1,25%	1,56%	1,84%	6,64%	6,64%
B	1,98%	2,88%	3,60%	7,69%	11,53%	18,98%	31,91%
CCC	2,99%	5,78%	9,52%	30,22%	31,17%	N/A	N/A

Table 4a – Average cumulative default rates for various credit-risk classes.
Defaults and issues from 1971 to 1987 (from: Altman, 1989)

S&P's original rating	**Average price after default**	**Number of observations**
AAA	78,67%	5
AA	79,29%	13
A	45,90%	19
BBB	45,30%	22
BB	35,71%	13
B	42,56%	64
CCC	41,15%	12

Table 4b – Average recovery rates for various credit-risk classes.
Defaults and issues from 1971 to 1987 (from: Altman, 1989)

Manufacturers of branded consumer products and pharmaceutical companies may have strong fundamentals as well; the latter usually benefit from high operating margins and high returns on total capital. Notice that newly issued investment grade bonds are very unlikely to go from heaven to hell abruptly, i.e. to default overnight.

REMARK.

The average cumulative default rates of Table 4a were estimated by using a method carried over from **actuarial sciences**. When examining a credit-risk class, say the AA, the historical time period 1971-1987 was considered and each year under examination was matched with a **cohort**, i.e. a pool of AA corporate bonds issued in that year. In contrast, when forming their cohorts, international credit-rating agencies consider bond issuers with a given rating on a given calendar date regardless of original rating and/or time since issue. The size of each cohort was represented by the **total face value** of the constituent bonds. Alternatively, it could have been represented by the **number of issuers**. The size of each cohort decreases as time goes by, since some bonds default and the principals of the non defaulted bonds are sooner or later repaid. Therefore, bond redemptions were taken into account when tracking the evolution of each cohort.

First of all, the yearly **marginal default rates** $mdr_1, mdr_2, \ldots, mdr_{10}$ were estimated for each cohort in accordance with the equation

$$mdr_t = \frac{\text{face value of bonds defaulting in year } t \text{ since issue}}{\text{size of the cohort at the start of year } t}$$

A weighted average $\overline{mdr_t}$ of each yearly marginal default rate mdr_t was then computed across the different cohorts, the weights being the ratios between the size of each cohort and the total size of all cohorts, both measured at the start of year t since issue.

Finally, the **average cumulative default rates** $\overline{cdr_1}, \overline{cdr_2}, \ldots, \overline{cdr_{10}}$ were estimated in accordance with the equation

$$\overline{cdr_t} = 1 - \prod_{k=1}^{t} \left(1 - \overline{mdr_k}\right)$$

which can be rewritten as

$$\overline{cdr_t} = \overline{mdr_1} + \overline{mdr_2}\left(1 - \overline{mdr_1}\right) + \overline{mdr_3}\left(1 - \overline{mdr_1}\right)\left(1 - \overline{mdr_2}\right) + \cdots + \overline{mdr_t} \prod_{k=1}^{t-1} \left(1 - \overline{mdr_k}\right)$$

Note that $\overline{mdr_1}$ and $\left(1 - \overline{mdr_1}\right)$ are the default and survival rates of the average cohort in the 1st year since issue, $\overline{mdr_2}\left(1 - \overline{mdr_1}\right)$ and $\left(1 - \overline{mdr_1}\right)\left(1 - \overline{mdr_2}\right)$ are the default and survival rates of the average cohort in the 2nd year since issue, whereas $\overline{mdr_3}\left(1 - \overline{mdr_1}\right)\left(1 - \overline{mdr_2}\right)$ is the default

rate of the average cohort in the 3rd year since issue. As a consequence, $\overline{cdr_t}$ is the default rate of the average cohort of AA corporate bonds in the first *t* years since issue.
Each recovery rate of Table 4b is an average of the market prices of some defaulting bonds. Recovery rates appeared to be unaffected by the age of the bond issue.

REMARK.
International credit-rating agencies are concerned with countries and their governments too. According to Standard & Poor's, the present long term rating is **AAA** for Australia, Canada, Germany, and Switzerland, **AA**+ for Austria and US, **AA** for France and UK, **A**+ for Israel and Japan, **A**– for Saudi Arabia and Spain, **BBB** for Italy, **BBB**– for Portugal. A low **sovereign** rating may be a ceiling for corporate ratings.

Credit rating is both an **art** and a **science**, as frequent resort is made to subjective judgement. Each company is assessed by more financial analysts, expert in the industry and region, who make use of heuristic rating guidelines and perform both a **business risk** analysis and a **financial risk** one by eliciting relevant information from managers. The former sets out from **business fundamentals** (i.e. country, industry structure, competitive strategy and financial plan, management character and commitment) to assess the presence and driving factors of a **competitive advantage**. The latter is based on **accounting ratios**, which are derived from past, *interim*, and *pro forma* financial statements and then compared to historical, industry-specific benchmarks. More precisely, use is made of *pro forma* financial statements whenever

- a new bond issue is to be rated;
- a **worst case** analysis, called a **stress test**, is first performed by defining a very adverse scenario, possibly on the basis of some historical event, and then by appraising the impact on the financial statements.

When using accounting ratios, a distinction should be drawn between short term **liquidity** and medium term **sustainability**, i.e. an ability to repay debt in the short and medium term. As explained in Benninga-Sarig (1997, chapt. 11), the former is measured by **liquidity** ratios, e.g.

- **current ratio**=current assets/current liabilities
- **acid test**=(cash & equivalents+other marketable securities+receivables)/current liabilities

whereas the latter is measured by **coverage** ratios, e.g.

- EBIT/interest expense
- cash from operations/interest expense

profitability ratios, e.g.

- **ROS**=EBIT/sales

- **ROI**=EBIT/mean total capital
- **ROE**=net income/mean equity capital

and **capital structure** ratios, e.g.

- total debt/total capital
- equity capital/(net fixed assets–property loans)

Coverage is the most important issue, unless there is a lack of liquidity.

International credit-rating agencies regularly publish some breakdown, by industry and credit-risk class, of the **median** values of selected accounting ratios; the grand summary of Table 5 can only be used for a statistical purpose. Table 5 rests on a sample of about 1.000 industrial companies; its entries may have been adjusted. Most AAA industrial companies are cash-rich pharmaceutical companies. The accounting ratio analysis may be complemented by a **cash flow** analysis, first dealing with the sources and uses of funds and then with the changes in liquidity and net working capital.

The lead financial analyst reports to a rating committee of, say, 5-7 members, who vote on his/her recommendation. Both all internal deliberations and the identities of all voting members are kept confidential. An appeal against the rating can be made before its publication on the basis of new or additional information.

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
EBIT/interest expense	23,8	19,5	8,0	4,7	2,5	1,2	0,4
EBITDA/interest expense	25,5	24,6	10,2	6,5	3,5	1,9	0,9
(Cash from operations–investment)/total debt (%)	127,6	44,5	25,0	17,3	8,3	2,8	–2,1
Total debt/EBITDA	0,4	0,9	1,6	2,2	3,5	5,3	7,9
EBIT/mean total capital (%)	27,6	27,0	17,5	13,4	11,3	8,7	3,2
Total debt/total capital (%)	12,4	28,3	37,5	42,5	53,7	75,9	113,5

Table 5 – Key accounting ratios by credit-risk class; three-year medians of industrial companies: 2002 to 2004 (adapted from: *Corporate Ratings Criteria,* Standard & Poor's, 2006)

Standard & Poor's financial analysts generally concentrate on one or two industries, covering all credit-risk classes. When monitoring a company, Standard & Poor's financial analysts meet its top managers once a year at least and hence get to know them. As a consequence, they can compare financial plans and financial statements over time, detecting deviations as well updates and seeking to understand the reasons why. In doing so, they can realise whether business plans tend to be either thoughtful and realistic or somewhat shallow and wishful. Ratings are reviewed in response to important financial transactions or unexpected developments.

Following Altman (1968) and its *z*-score model based on **discriminant analysis** (see Exercise **43**), practitioners may build their scoring model that turn the different financial ratios into a composite score. Those models prove to be useful, when it comes to assess the **creditworthiness** of unrated companies or anticipate rating changes by credit-rating agencies. Moreover, the stocks of companies with dismal prospects are eligible for short sales. Credit scoring models can be based on **hazard analysis** as well.

When drawing Table 6 below, an initial outlay in each credit-risk class was considered and the time evolution of its accumulation excess was traced. US Treasury bonds were the benchmark; their cohort included the same maturities with the same initial weights. All bonds, whether corporate or Treasury, were supposedly bought at issue and held until expiry, each coupon being reinvested in the corresponding cohort; moreover, each defaulting loan was supposedly sold, the receipt being reinvested in the corresponding surviving cohort. The average **credit spreads** over the period 1971-1987 were 0,47% (AAA), 0,81% (AA), 1,08% (A), 1,77% (BBB), 3,05% (BB), 4,09% (B), and 7,07% (CCC).

If the default probabilities implicit in credit spreads and bond prices (see Exercise **41** and Exercise **42**) had been the same as the actual default rates, all default losses would have been precisely offset by the excess returns on non defaulted corporate bonds. Therefore, all accumulation excesses would have vanished in the long run. This was not the case of Table 6, where credit spreads did reflect more than forecast default losses and included a **credit-risk premium** as well as a **tax compensation**. In other words, the basic version of the principle of compensation among risks, a foundation of **actuarial sciences**, proved to be suitably extended. Keep in mind that interest on US corporate bonds is subject to both federal and state tax, whereas interest on US Treasury bonds is exempt from state tax.

Years after issuance	**Bond rating at issuance**						
	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
1	0,45%	0,76%	1,04%	1,71%	3,26%	3,82%	5,19%
2	1,00%	1,68%	2,23%	3,66%	6,84%	8,61%	11,74%
3	1,65%	2,43%	3,67%	6,09%	11,29%	14,60%	20,62%
5	3,44%	5,15%	7,82%	12,50%	24,19%	21,60%	15,61%
7	5,98%	9,49%	13,66%	22,86%	35,85%	33,65%	N/A
10	12,45%	20,28%	28,85%	45,77%	76,37%	44,67%	N/A

Table 6 – Accumulation excesses attained by corporate bonds over US Treasury bonds.
Defaults and issues from 1971 to 1987 (from: Altman, 1989)

Exercise 41.

Consider 2 **imaginary** and **large** portfolios that include many different **zero coupon bonds**, all with an original term to maturity of 1 year. The aggregate face value of each portfolio is €10.000.000, with the yearly yield to maturity of the former (latter) being 4% (4,25%). Suppose that when a default occurs, the face value of the defaulted bond is paid at expiry by a recovery rate of 50%. All bonds of the former (latter) portfolio belong to the credit-risk class AAA (BBB); AAA bonds bear no risk of default. Find

a) the current price of each bond portfolio;

b) the credit spread *sp* of BBB bonds with an original term to maturity of 1 year;

c) the default probability of BBB bonds in the first year after issue.

Solution.

Let time be measured in years and time 0 be the appraisal date.

a) The current price of the former portfolio is $P_{AAA} = 10.000.000 * 1{,}04^{-1} = 9.615.384{,}62$, whereas the current price of the latter is $P_{BBB} = 1.000.000 * 1{,}0425^{-1} = 9.592.326{,}14$.

b) Since AAA bonds bear no risk of default, the 1 year spot rate $i_{0;1}$ on a safe transaction is equal to 4% too. The credit spread sought is

$$sp = 4{,}25\% - 4{,}00\% = 0{,}25\%$$

c) In contrast, BBB bonds bear a risk of default; nonetheless, if the number of issuers and **industries** is large enough, **credit risks** are likely to **compensate**. Indeed, if numbers are **grand**, with past and future being the same in **probabilistic** terms, which could not be true in practice, then the actual default rate will be close to the default probability, in turn close to the average default rate measured over, say, the past 7 year term. As a consequence, the actual receipt at expiry will be approximately worth

$$10.000.000 * (1 - \pi_{BBB}) + 5.000.000 * \pi_{BBB}$$

where π_{BBB} is the default probability of BBB bonds in the first year after issue. Therefore, arbitrage is ruled out if

$$(10.000.000(1 - \pi_{BBB}) + 5.000.000\pi_{BBB}) * 1{,}04^{-1} = 9.592.326{,}14$$

from which it follows that $5.000.000\pi_{BBB} = 23.980{,}81$ and hence $\pi_{BBB} = 0{,}480\%$.

Exercise 42.

Consider the setting of Exercise **41** as well as 2 **imaginary** and **large** bond portfolios that include many different **coupon bonds**, all with an original term to maturity of 2 years. The aggregate face value of each portfolio is €10.000.000, with the coupon rate and the yearly yield to maturity of the former (latter) being the same and equal to 4% (4,50%). Suppose that when a default occurs, the remaining coupons of the defaulted bond are not paid, with the recovery rate being 50%. All bonds of the former (latter) portfolio belong to the credit-risk class AAA (BBB); AAA bonds bear no risk of default. Find

a) the current price of each bond portfolio;

b) the credit spread *sp* of BBB bonds with an original term to maturity of 2 years;

c) the default probability of BBB bonds in the second year after issue.

Solution.

Let time be measured in years and time 0 be the appraisal date.

a) Since coupon rate and yield to maturity are the same, the current price of the former portfolio is $P_{AAA} = 10.000.000$, whereas the current price of the latter is $P_{BBB} = 10.000.000$.

b) Since AAA bonds bear no risk of default, the spot rates $i_{0;1}$ and $i_{0;2}$ on a safe transaction are equal to 4% too. The credit spread sought is

$$sp = 4{,}50\% - 4{,}00\% = 0{,}50\%$$

being also the solution to the equation $100 = 4{,}50 * \left(1 + i_{0;1} + sp\right)^{-1} + 104{,}50 * \left(1 + i_{0;2} + sp\right)^{-2}$.

c) In contrast, BBB bonds bear a risk of default; nonetheless, if the number of issuers and **industries** is large enough, **credit risks** are likely to **compensate**. Indeed, if numbers are **grand**, with past and future being the same in **probabilistic** terms, which could not be true in practice, then each actual default rate will be close to the corresponding default probability, in turn close to the average default rate measured over, say, the past 7 year term. As a consequence, the actual receipts will be approximately worth

$$450.000 * 0{,}9952 + 5.000.000 * 0{,}0048 = 471.840{,}0$$

after 1 year and

$$10.450.000 * \left(1 - \pi_{BBB;1} - \pi_{BBB;2}\right) + 5.000.000 * \pi_{BBB;2} = 10.399.840 - 5.450.000\,\pi_{BBB;2}$$

after 2 years, where $\pi_{BBB;t}$ is the default probability of BBB bonds in the *t*-th year after issue and $\pi_{BBB;1} = 0{,}480\%$. Therefore, arbitrage is ruled out if

$$10.000.000 = 471.840 * 1{,}04^{-1} + \left(10.399.840 - 5.450.000\pi_{BBB;2}\right) * 1{,}04^{-2}$$

from which it follows that $5.450.000\pi_{BBB;2} = 74.553{,}60$ and hence $\pi_{BBB;2} = 1{,}368\%$.

REMARK.

The default probabilities $\pi_{BBB;1}$ and $\frac{\pi_{BBB;2}}{1-\pi_{BBB;1}}$ can be compared with the historical weighted averages $\overline{mdr_1}$ and $\overline{mdr_2}$ of the yearly **marginal default rates** of BBB bonds. More generally, the default probability $\frac{\pi_{BBB;t}}{1-\left(\pi_{BBB;1}+\pi_{BBB;2}+\cdots+\pi_{BBB;t\text{-}1}\right)}$ can be compared with $\overline{mdr_t}$. The above calculations apply to individual bonds as well, which are tacitly supposed to be held in large, well diversified and possibly heterogeneous portfolios.

REMARK.

A **retrospective analysis** of defaults can rest on either statistical data or models. Table 4 belongs to the former class, which includes similar tables drawn up by credit-rating agencies. According to those tables, the first 2 years of life are very critical for all bonds with such a low credit rating as CCC; moreover, the rating drift entails that the average marginal default rates of speculative grade bonds don't increase, as time since issue goes by. The scoring models based on **discriminant** or **hazard analysis** belong to the latter class.

When performing a **prospective analysis** of yearly default rates, one has to keep in mind that any default probability implicit in bond prices depends on the business cycle, being generally greater than an empirical average marginal rate, especially in case of a recession. This would follow from the difficulties faced by bond managers in achieving an effective portfolio diversification; furthermore, a less effective compensation of credit risks does occur in a recession owing to knock-on effects on defaults. Nonetheless, a larger default probability is matched by a higher yield to maturity and, possibly, by **excess returns**.

Exercise 43.

An Italian company listed on the Milan Stock Exchange produces loudspeakers. In 2011 it had proceeds of sales of 27.693, earnings before interest and taxes of 5.658, a working capital of 11.331, total assets of 20.270, retained earnings of 7.575, a book value of total debt of 2.531, and an average market value of equity of 38.000 (data are expressed in $10^3€$). Check that the company was likely to remain solvent in 2012, as it actually occurred.

Solution.

Failing companies and continuing entities display different accounting and financial ratios. According to Altman (1968), accounting and financial ratios can be turned into the following composite *z*-score

$$z = 1{,}2x_1 + 1{,}4x_2 + 3{,}3x_3 + 0{,}6x_4 + 0{,}999x_5$$

where x_1 is the **liquidity** ratio between working capital (=current assets–current liabilities) and total assets, x_2 is the ratio between retained earnings and total assets, x_3 is the ratio between EBIT and total assets, x_4 is the **capital structure** ratio between market value of equity and book value of total debt, whereas x_5 is the ratio between proceeds of sales and total assets. If

- $2{,}99 < z$, the firm under examination is likely to remain solvent;
- $2{,}675 < z \le 2{,}99$, one should be on alert;
- $1{,}81 < z \le 2{,}675$, the firm under examination is likely to fail within a year;
- $z \le 1{,}81$, the firm under examination is very likely to fail within a year.

If we stretch such scoring model a little bit and apply it to the Italian company, we get

$$x_1 = 0{,}559;\ x_2 = 0{,}374;\ x_3 = 0{,}279;\ x_4 = 15{,}014;\ x_5 = 1{,}366$$

so that $z = 12{,}49$. Therefore, the Italian company was likely to remain solvent.

REMARK.

Altman (1968) made use of **discriminant analysis** to derive his scoring model. The data sample included 66 manufacturing companies listed in the US; 33 of such companies went bankrupt during the period 1946-1965, whereas the remaining 33 were still in existence in 1966.

Type I and type II errors were small: only 6% (3%) of the bankrupt (solvent) companies were incorrectly classified. Altogether, the scoring model proved reliable up to 2 years before bankruptcy. When data from 2 years before bankruptcy were used, type I and type II errors were 28% and 6%, respectively. Moreover, when a different data sample was considered, the above scoring model proved accurate again.

Finally, the 5 ratios were averaged across all bankrupt companies for 5 years before bankruptcy. All ratios displayed a deteriorating trend, with their largest decrease occurring 2 years before bankruptcy in 3 out of 5 cases.

4.4. Securitization of non marketable credits

A securitization process turns some non marketable credits into marketable securities. It is applied to a large pool of non marketable credits with multiyear repayment and similar features, such as residential or commercial mortgages, consumer credits, leases as well as non performing loans. Companies (public bodies) carry out their securitizations too, e.g. of trade receivables on a revolving basis (of social security contributions). The first securitizations took place in the US in the 1970s.

REMARK.
The presentation in this section is consistent with the case of Italy, where securitizations are regulated by the law No. 130 30/4/1999, as amended by the law No. 80 14/5/2005.

Non marketable credits are sold **without recourse** by an **originator** to a **special purpose vehicle**, an authorized entity that is solely devoted to one or more securitization processes. For instance, residential mortgage loans are sold by an Italian bank without recourse; if such non marketable credits are good quality, their price will be greater than their face value. The special purpose vehicle has little equity, no employees, and outsources all services; it issues fixed or floating rate bonds on primary markets, usually placing them with institutional investors. In this manner, it raises money to fund the purchase. A **trustee** acts on behalf of bondholders. Bonds may repay their face value in instalments; in any case, they are split into a few tranches with different creditworthiness, usually rated by one or more international credit-rating agencies. Whenever bonds are publicly placed, credit rating is mandatory; in principle, it should provide reliable information to all potential subscribers. The worse the creditworthiness is, the higher the degree of subordination and the coupon rate; bonds of the senior (most junior) tranche have the best (worst) creditworthiness and are repaid first (last). The face value of bonds diminishes owing to defaults; the most junior (senior) tranche is affected first (last).
A **servicer**, who is usually the originator, monitors the pool of residential mortgages, collects interest and capital repayments from borrowers, and transfers them to the special purpose vehicle, managing all impairments and non performing loans. In this manner, the bank doesn’t lose the relationship with its borrowers and is paid commissions on a regular basis. The special

purpose vehicle uses such repayments to reward the various tranches of bonds in accordance with their priority order.
As previously mentioned, the default risk is borne by the special purpose vehicle and hence all bondholders. The pool of residential mortgages is well diversified by geographic location. Moreover, it is granular, each individual borrower having a small weight; for instance, it might include more than 10.000 residential mortgages. Nevertheless, credit is enhanced in different respects. First of all, bonds are split into a few tranches; as the senior tranche is sheltered by more subordinated junior tranches, it is likely to be AAA (Aaa) rated. Moreover, default risk can be mitigated by **overcollateralisation** or additional guarantees. Overcollateralisation occurs when the face value of the residential mortgage loans is appropriately larger than that of the bonds. Additional guarantees are generally provided by other banks, through letters of credit, or insurance companies, e.g. through surety bonds, against the payment of regular commissions or premia. Finally, the most junior tranche is typically retained by the originator.

REMARK.
When structuring or rating the different tranches, one needs to estimate the time pattern of the expected loss on the pool of residential mortgages. The securitization under scrutiny is **traditional**, as the pool of residential mortgages is granular. As for default risk, the individual residential mortgages can be considered as independent of each other. Therefore, default probabilities and recovery rates can be estimated by applying **actuarial methods** to historical data; more precisely, the empirical weighted averages of yearly marginal default rates are used as proxies for default probabilities. The expected loss in the t-th year since issue is equal to the corresponding default probability multiplied by the **loss given default**, which depends on the **exposure** and the recovery rate.
In contrast, if the securitization were **innovative**, the pool could be made up of commercial loans provided to fewer than 200-300 borrowers. As the individual commercial loans are dependent on each other, default correlations need to be taken into account. As a consequence, both financial calculus and estimation procedures become more involved.

Such a securitization process is complex and very expensive; therefore, the pool of residential mortgages is large and the process is run by an **arranger** such as a merchant, investment, or universal bank.

REMARK.

Initial expenditures are very large and derive from the mandate to the arranger, the establishment of the special purpose vehicle, credit rating, and bonds placement. If an Italian company carried out a securitization with a term of 5 years and a **revolving base** made up of trade receivables with nominal value of €100 million, the above-mentioned expenditures might respectively amount to 0,25%, 0,10%, 0,02%, and 0,25% of the capital initially raised. Keep in mind that in the first stage of a revolving securitization, the special purpose vehicle uses a portion of receipts to buy additional non marketable credits.

The arranger carryies out several tasks, working together with the originator as well as auditors, lawyers and tax consultants. As a consequence, the residential mortgages to securitize are picked out; some possible structures of the transaction are determined, i.e. all tranches and their features; third parties, providing additional guarantees, are selected on the basis of their reputation. Once the residential mortgages to securitize have been determined, the arranger carries out due diligence on them, checking, e.g., the rights of the bank, the values of the morgtgaged properties, and whether the beneficiaries of the insurance policies against damages to the mortgaged properties can be replaced. Auditors must certify that such residential mortgages are suited for the securitization.

The arranger submits the pool of residential mortgages and the different transaction options to credit-rating agencies; on negotiating possible changes, a preliminary agreement is reached, consistent with all rating targets. As a consequence, the receipts of the originator can be forecast with greater accuracy; they derive from bonds placement and servicing. Bonds are officially rated only after they have been issued by the special purpose vehicle. The arranger usually runs the bonds placement.

REMARK.

The securitization of residential mortgages brings about several benefits to the Italian bank under examination: funding is diversified and bargaining power is enhanced; liquidity is expanded and invested capital is released; both the matching of assets and liabilities and the management of interest rate risk are facilitated; both credit risk and the cost of funding are brought down; the need of regulatory capital is temporarily lessened; visibility in the financial markets is increased.

Covered bonds are directly issued by a **bank** or **credit union** with an appropriate regulatory capital. They offer a twofold protection. First of all, they are backed by a separate **cover pool** of high quality and non marketable credits, i.e. mortgage or public sector loans from the bank (or another bank with appropriate regulatory capital). Moreover, and in contrast with a securitization, covered bondholders have full recourse to the bank. Before issuing the covered bonds, the bank sells the cover pool to a **special purpose vehicle**, which guarantees the bond issue in return. As a consequence, if the bond issuer became insolvent, only covered bondholders would have recourse to the separate cover pool. The bank has the obligation to ensure that the present (face) value of the cover pool is consistently equal to or greater than the present (face) value of the covered bonds. If defaults in the cover portfolio were higher than anticipated, the bank should transfer additional non marketable credits to the special purpose vehicle. The transaction is monitored by an audit firm, which checks whether regulations are fulfilled and the cover pool is adequate. Therefore, and in contrast with a securitization, default risk is borne by the bank. As the separate cover pool is made up of high quality and non marketable credits, covered bonds are considered **investment grade** by international credit-rating agencies and rated AAA or Aaa in many instances. Cover bonds typically repay their face value at expiry.

The most recent global financial crisis

The securitization of residential mortgages played an important role in the US financial crisis that occurred in the years 2007 and 2008 and triggered a **global financial crisis**.

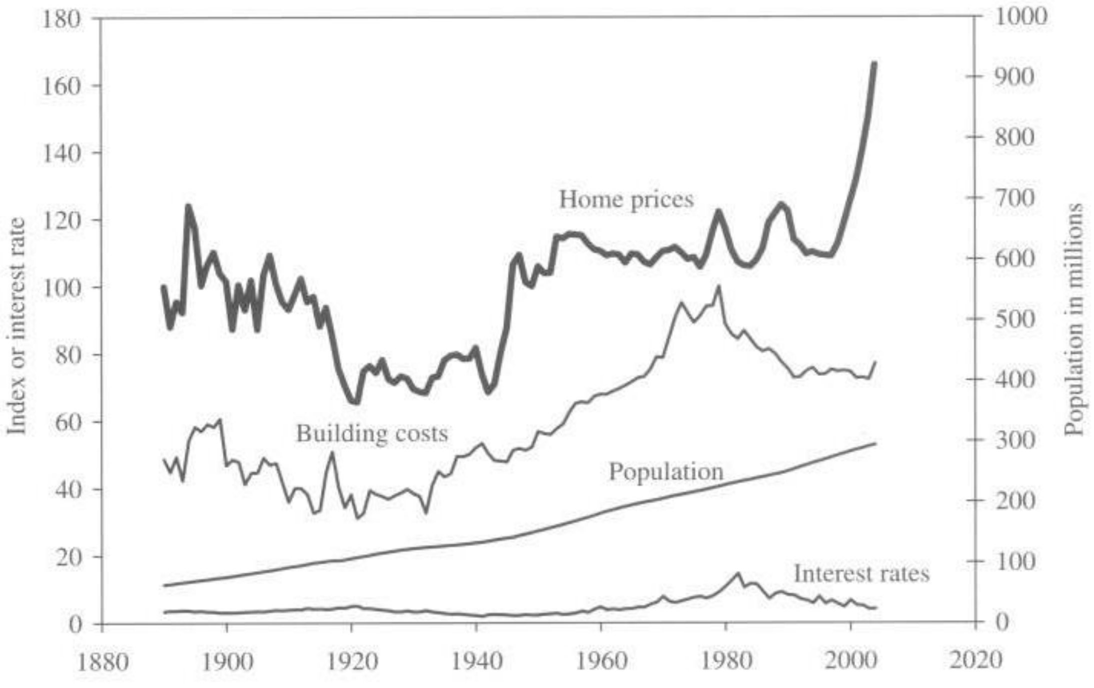


Figure 1 – Real housing prices (1890=100), real building costs (1979=100), population (in millions), and long-term interest rates, US 1890-2004 (from: Shiller, 2005, chapt. 2)

Since the 1990s, market interest rates were low in the US and subprime mortgage loans were provided to low income and poor credit standing borrowers, previously excluded from the credit market. Mortgages loans were liquid owing to Fannie Mae (Federal National Mortgage Association) and Freddie Mac (Federal Home Loan Mortgage Corporation), 2 government-sponsored enterprises that carried out their securitization; the bonds issued were guaranteed against default risk. US housing prices started growing at the end of the 1990s, almost tripling in the decade 1997-2006. Notably, US stock indexes had tripled in the years 1994-2000. Indeed, housing prices in 13 developed countries have shown a tendency to peak, on average, a couple of years after stock prices. The historical pattern in US housing prices is displayed in Figure 1 above.

Although only conforming residential mortgages were considered at first, subprime mortgages were securitized later by other financial intermediaries. In the latter case, the bonds issued weren't guaranteed against default risk, which was transferred from lenders to bondholders. Owing to the swift and wide diffusion of the latter securitizations, there was a considerable growth in subprime mortgages between 2000 and 2006. Unfortunately, many loan applications were poorly managed by **mortgage brokers**, i.e. non-bank intermediaries: documentation wasn't checked (appropriately); the ratio between amount lent and home value was often too high. Moreover, many subprime mortgages charged low interest rates for the first 2 or 3 years and market adjustable rates subsequently, e.g. the US Treasury bill rate plus 3%, which were not in line with low income and naïve borrowers. They agreed on such contractual terms in the belief they would be able to refinance more favourably later on. In spite of this, when securitizing subprime mortgages, the senior tranche was frequently rated AAA or Aaa by international credit-rating agencies. With the benefit of hindsight, we can claim that possible default rates were understated by international credit-rating agencies, as use was made of historical data from a very favourable historical period and about safer residential mortages loans. Indeed, no allowance was made for the case of declining housing prices. At the same time, some counterparts of credit default swaps such as AIG (an acronym of American International Group) sold too much insurance against default risk, not in line with the available capital. The above-mentioned special purpose vehicles entered into credit default swaps to enhance credit. In 2005 and 2006 about $1,2 trillion were likely to be lent through subprime mortgages, which were securitized by a proportion of 80%.

US housing prices peaked in mid-2006 and entered a stage of decline, matched by higher adjustable rates for subprime mortgages, which had been rising since 2004. US housing prices would shrink by nearly one third by mid-2009. Refinancing became much more difficult; moreover, the house could be worth less than the outstanding loan. As delinquencies from naïve or opportunistic borrowers soared, foreclosures kept up so that a large number of houses came on the market, driving their prices further down and thus feeding a vicious circle. Consequently, also the above-mentioned AAA or Aaa tranches suffered from serious losses in 2007 and 2008 so that the capital of important banks such as Citigroup and Wachovia Bank was badly eroded. It was the onset of the US financial crisis, which peaked in September 2008, when Fannie Mae and Freddie Mac were placed into conservatorship, Merrill Lynch, on the verge of bankruptcy, was sold to Bank of America, Lehman Brothers, founded in the 1850s, went bust, and AIG was bailed out by the US government. This led to a global credit crunch so that economic growth slowed worldwide and international trade declined.

4.5. Active bond management

When it comes to **active bond management**, a distinction has to be made between countries and credit-risk classes, of which there are 10 according to such credit-rating agencies as Fitch Ratings, Moody's Investors Service and Standard & Poor's. In principle, there are 10 credit-risk classes in each country, each one being daily represented by a specific **yield curve**, whereby any relevant **term** on the horizontal axis is matched by a **yield to maturity** on the vertical axis. As explained below, **government bonds** (e.g., zero coupon, fixed rate and floating rate ones) and their yields to maturity mirror the macroeconomic outlook of a country so that **Treasury** yields are a daily benchmark. If their creditworthiness is AAA or AA according to Fitch Ratings and Standard & Poor's and Aaa or Aa according to Moody's, their default risk is negligible and the lowest possible. Roughly speaking, there are 3 key driving factors of a bond yield to maturity: **1**) the maturity, **2**) the country and its **macroeconomic outlook**, **3**) the issuer's creditworthiness, in turn depending on an **industry's outlook** as well as on some company specific and partially **diversifiable** risk factors. For any given maturity, the larger the nominal credit spread over the corresponding government bonds, i.e. the simplest possible measure of the credit-risk premium, the lower is the issuer's creditworthiness.

The scientific works on **bond mutual funds** are far fewer than those on equity mutual funds. According to those empirical analyses, the overall historical performance of bond mutual funds has been generally poor with no evidence of performance persistence. However, the gist of active bond management lies in making better forecasts than the consensus ones implicit in the various yield curves. Indeed, a Treasury yield curve changes its shape over time, generally in line with the business cycle, being determined by consensus forecasts on the economy as well as imbalances between supply and demand. In principle, consensus forecasts should concern the macroeconomic outlook of a country, as defined in the tables printed in *The Economist* magazine, i.e. by a GDP growth rate, an inflation rate, an unemployment rate, an exchange rate, a trade balance as well as by the 2 ratios current account balance-GDP and budget balance-GDP. The consensus expectations on the **monetary policy** of a Central bank should be mirrored in the first part of the Treasury yield curve, where maturities are lower than 2 years; moreover, if expansion and more inflation (recession and less inflation) are expected in the short-medium run, as is before a trough (peak) of the business cycle, the Treasury yield curve should slope upward (downward). As bondholders are risk averse, the Treasury yield curve is typically upward-sloping; nonetheless, it evolves over time so that short term yields vary much more than long term ones.

As for corporate bonds, the **credit spreads** depend on the business cycle and do widen (narrow) in a downturn (upturn), when defaults are more (less) frequent and default rates are higher (lower). The relative yield spread between 10-year BBB and AAA bonds provides a clue to overall risk perception.

According to some **principal component analyses**, the variability in Treasury yields to maturity, say 95% of it, is mostly explained by 3 common factors, which are conventionally interpreted as a parallel shift, a change of slope, and a change of curvature in the Treasury yield curve.

REMARK.

As explained in Golub-Tilman (2000, chapt. 3) and the references therein, a **principal component** analysis of bond excess returns may feature

- reference to zero-coupon bonds issued by US Treasury or created by brokers/dealers, who split Treasury coupon bonds into individual principal and interest payments and resell them separately in the secondary market (Separate Trading of Registered Interest and Principal Securities, introduced by US Treasury in 1985). For instance, an **excess return** vector $er_{n;1}$ with respect to the overnight repurchase agreement rate is **weekly** sampled over a 5-year term so that a **real** and **symmetric** variance-covariance matrix $\Sigma_{n;n}$ is computed. Use of **exponential weighting** is viable;
- a change of variables $pc_{n;1} = \Omega_{n;n} er_{n;1}$, where Ω is an orthogonal matrix. If the n eigenvalues of Σ are distinct and positive, i.e. such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$, the columns of $\Omega^T = \Omega^{-1}$ are the corresponding **orthogonal** eigenvectors of Σ, each normalised so that the sum of its squared coefficients is 1. As a consequence, a vector $pc_{n;1}$ of orthogonal (and hence incorrelated) **principal components** is obtained, with their variance-covariance matrix being the **diagonal** matrix $\Omega\Sigma\Omega^T$ so that the sum of

their variances $\lambda_1 + \lambda_2 + \ldots + \lambda_n$ is equal to $\sigma_1^2 + \sigma_2^2 + \ldots + \sigma_n^2$. Only 2-3 principal components usually do matter and explain on average, say, 85%, 7%, and 3% of the variability in excess returns, namely

$$\frac{\lambda_1}{\lambda_1 + \lambda_2 + \ldots + \lambda_n} + \frac{\lambda_2}{\lambda_1 + \lambda_2 + \ldots + \lambda_n} + \frac{\lambda_3}{\lambda_1 + \lambda_2 + \ldots + \lambda_n} = 0{,}85 + 0{,}07 + 0{,}03 = 0{,}95$$

The financial interpretation of those common factors is the above-mentioned one. We have

$$er_k = \omega_{1;k}\, pc_1 + \omega_{2;k}\, pc_2 + \omega_{3;k}\, pc_3$$

where the coefficients $\omega_{1;1} \cong \omega_{1;2} \cong \cdots \cong \omega_{1;n}$ of the 1st eigenvector represent a parallel shift, whereas the coefficients $\omega_{2;1} \geq \omega_{2;2} \geq \cdots \geq \omega_{2;n}$ (or $\omega_{2;1} \leq \omega_{2;2} \leq \cdots \leq \omega_{2;n}$) of the 2nd eigenvector represent a clockwise (counterclockwise) nonrigid rotation. It is readily ascertained that the variance of each weekly excess return is mostly explained by the 3 common factors;

- reference to some coupon bonds issued by US Treasury. It is likely to check **in sample** that the weekly excess returns on those bonds are mostly affected by the 3 above-mentioned common factors.

A principal component analysis can also be performed on daily, weekly, or monthly changes in either the Treasury yields to maturity or the key spot rates of the money market. The **persistence** of the relationship can be ascertained by splitting the data sample into subperiods. Remarkably, the weights of the first common factor pc_1 and volatilities may display a similar dependence on maturities.

Active bond management is essentially concerned with anticipating the changes in **Treasury yields** and **credit spreads** as well as with identifying any **temporary mispricing** of bonds and bond sectors.

As for Treasury yields, a bond manager should forecast their (short-term) evolution, earlier or better than is anticipated by the market, a challenging if not daunting task indeed.

For instance, if the Treasury yield curve slopes upward and if he expects it to remain largely unchanged, he will ride it a little bit, swapping cautiously shorter maturities for longer ones, thus taking some more **interest rate risk** in return for higher yields to maturity.

As portrayed in Figure 2 below, he may also expect all yields to increase (decrease) owing to a parallel upward (downward) shift of the Treasury yield curve, accompanied by a flattening (steepening) twist, i.e. a falling (rising) spread between long and short term yields. In that circumstance, the bond manager will shorten (lengthen) the portfolio duration appropriately; indeed, if bonds have a small (fairly large) financial duration, their prices are not so (considerably and favourably) affected by such a change.

REMARK.

As reported in Jones (1991), the 2 above-mentioned shifts account for 91,6% of actual yields on US Treasury securities. Moreover, they are likely to be matched by large yield changes, with a 1% parallel upward (downward) shift being consistent with, say, a 0,25% decrease (increase) of the spread between between long and short term yields. However, the effects of the flattening (steepening) twist may be stronger than those of the parallel upward (downard) shift.

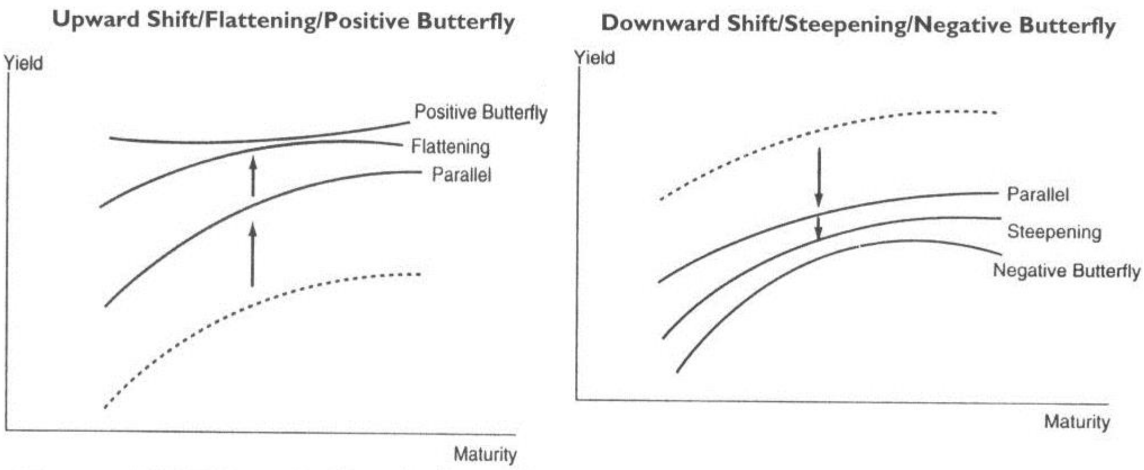


Figure 2 – Typical combinations of shifts of the US Treasury yield curve (from: Jones, 1991)

REMARK.

As for Treasury zero coupon bonds, it can be ascertained that if the above-mentioned expectations come true, a **barbell** (**bullet**) portfolio may outperform a **bullet** (**barbell**) one with the same financial duration; in other words, zero coupon bonds concentrated at 2 extreme maturities, a short and a long one, may do better than zero coupon bonds concentrated at one intermediate maturity, close to the financial duration. A parallel shift in the yield curve causes the barbell portfolio to slightly outperform the bullet one, as the former has a greater convexity than the latter. A flattening (steepening) twist generally favours the barbell (bullet) portfolio over the bullet (barbell) one; the barbell (bullet) portfolio includes (doesn't include) the long-maturity zero coupon bonds, which are most affected by the twist. A notable exception to the rule is that of a sizeable parallel downward shift along with a very mild steepening twist, with the former exerting a greater effect than the latter.

More generally, a bond manager may outline 3 different scenarios at least and assign them their own probabilities of occurrence; each scenario, be it a pessimistic, an intermediate, or an optimistic one, describes a peculiar pattern of the Treasury yield curve, say at the year end. He will be likely to strive for a compromise between risk and return, favouring a portfolio with an acceptable performance in each scenario, possibly not excellent in the optimistic scenario, though satisfactory in the pessimistic one.

As for corporate bonds, a bond manager will overweigh (underweigh) a bond sector, i.e. part of a credit-risk class, if he expects the corresponding credit spread to drop (soar) soon, being not in line with issuers' fundamentals and historical records, owing to unjustified pessimism (optimism) in his opinion. In principle, any credit spread bet should be consistent with the portfolio duration target. Finally, a bond manager will take advantage of the opportunity to swap some bonds held for some cheaper ones that have the same characteristics altogether. Moreover, he will pick some mispriced bonds in his opinion, thus benefiting from an attractive yield to maturity and/or a capital gain in the near future consequent on an upgrade by international credit-rating agencies.

REMARK.

According to historical records, investment-grade bonds are more likely to be downgraded than upgraded by credit-rating agencies. Moreover, the extra returns on mispriced bonds are generally much lower than those on mispriced stocks. Nonetheless, credit risk can be borne by buying and holding a large and well diversified portfolio of lower-rated bonds. According to Table 6, credit spreads might more than compensate for the default losses so that excess returns might ensue. However, if such bonds weren't properly selected owing to a poor fundamental analysis of bond issuers, default losses could be larger than expected. Moreover, short term capital losses could occur owing to a rise of credit spreads.

The interested reader might consult Farrell (1997, chapt. 14) for a thorough presentation of active bond management, also dealing with

- bonds that contain a call option for the issuer, who can redeem the bonds prior to expiry, generally paying a call premium;
- foreign bonds, that allow a wider bond portfolio diversification, and bear a currency risk. In that case, a bond manager has to decide whether to mitigate or hedge such a risk, by regularly entering into some appropriate derivative contracts.

5. Term structure of interest rates

5.1. Measurement of some spot rates (almost) free of credit risk

Whenever the credit rating of a government is AAA or AA according to Fitch Ratings and Standard & Poor's, and Aaa or Aa according to Moody's Investors Service, there are several rates of interest which come along with negligible **credit risk** under ordinary operational conditions, i.e.

- the yields to maturity on government bonds;
- the interbank rates;
- the swap rates as well as the rates implicit in some futures contracts;
- the rates of repurchase agreements.

Some **Euribor** and **Irs** rates on a particular day are reported in the 2 following tables.

ask rate (%)	3,159	3,209	3,251	3,610	3,805	3,879	3,928	3,963	3,978
term	1w	2w	3w	1m	2m	3m	6m	9m	1y

Table 7a –**Euribor** rates on 27/11/2008
(adapted from: il Sole 24 Ore, Friday 28/11/2008)

The Euro Interbank Offered Rate is a **yearly** rate of **simple** interest applicable to **unsecured** interbank loans in the €zone with a settlement lag of 2 business days and according to an **actual/360** day count convention. The €zone included 19 (11) countries in 2018 (1999). Those unsecured interbank loans are usually taken out by counterparts with a sound credit rating (AA or Aa in principle); they take the form of a deposit and have terms ranging from 1, 2, 3 weeks, or 1, 2 months, …, to 12 months. More precisely, each Euribor is a truncated mean, once calculated by Reuters, of the rates **posted** by a sample of 20 banks. Most transactions have a short term that does not exceed one month; each transaction is worth €1 million at least.

bid rate (%)	3,33	3,14	3,23	3,35	3,46	3,94	4,05	3,64	3,31
ask rate (%)	3,35	3,16	3,25	3,37	3,48	3,96	4,07	3,66	3,33
term	1y	2y	3y	4y	5y	10y	20y	30y	50y

Table 7b – **IRS** rates on € against **6 month** Euribor on 27/11/2008
(adapted from: il Sole 24 Ore, Friday 28/11/2008)

An Interest Rate Swap is a derivative contract, entered into by 2 counterparts, who agree on exchanging **half-yearly** payments **in arrears** for 1 or more years; both sequences of half-yearly payments, called legs by market practitioners, are a function of a notional capital, which is not exchanged. The effective date follows the fixing date by 2 business days. The **floating** leg is made up of half-yearly floating payments in arrears, each depending on the notional capital and the floating rate, the 6 month Euribor with settlement at the very beginning of the corresponding 6 month term. The **fixed** leg is made up of half-yearly fixed payments in arrears, each worth the notional capital multiplied by half of the fixed rate, the Irs rate agreed upon at trade time. More precisely, use is made of the bid (ask) rate, if the half-yearly fixed payments are made (received) by the financial intermediary; any bid-ask spread provides a reward for financial intermediation.

When it comes to tackle such financial problems as the measurement of the credit spread of a bond or the valuation of derivative contracts, a term structure is needed of **homogeneous** interest rates that are (almost) free of credit risk. Let $i_{t;T}$ be a **spot** rate of interest **per year**, charged on a loan that starts at time *t*, ends at time *T*, and is repayable with a lump sum. A **term structure** is a finite sequence of elements $\{i_{t;T}\}$ with *t* given, *T* free, such that the accumulation factors $1+i_{t;T}(T-t)$ and/or $(1+i_{t;T})^{T-t}$ do increase with the **term** $T-t$. When not stated otherwise, **settlement lags**, **commissions**, **fees**, and **taxes** are assumed away and 30-day months are considered throughout this section, in line with the 30/360 European day count convention.

If reference is made to an appropriate currency and the above-mentioned interest rates, 2 term structures (almost) free of credit risk can be measured every day, with the former (latter) concerning the money market (Treasury market). For any term, the former (latter) points out the yearly **spot** rate at which a financial intermediary (a government) with a sound credit rating (AA or Aa in principle) may borrow money. Implicit interbank rates are calculated by using the **bootstrap** method, whereas Treasury bonds are easier to treat owing to coupon stripping (see Exercise **47**).

REMARK.

Let $y_{t;T}$ the yield to maturity at time *t* of a Treasury bond that expires at time *T*. The **yield curve** $\{y_{t;T}\}$ is other than the **term structure** of interest rates $\{i_{t;T}\}$.

If derivative contracts are appraised by using the **money market** rates, as usually occurs in the business place, the resulting theoretical prices will rule out any arbitrage opportunities both for banks and other financial intermediaries, since worse terms are applied to the latter ones (lower rates on deposits and higher rates on loans).

The **interbank** market is supported by a telematic network linking all authorized operators. For each term shorter than a year, bid and ask rates are quoted every day; the former is applied to deposits, the latter to loans. Reference is usually made to the ask rate, which is greater than the bid one. Since the risk of default of a bank is larger than that of its National Treasury, each interbank ask rate is in theory larger than the corresponding spot rate (implicit in the prices) of Treasury bonds.

The term structure of interbank rates, i.e. of the **money market**, can be extended for several terms longer than 1 year by making use of swap rates. More specifically, if the **bootstrap** method is applied to the data in Table 7b, some implicit Euribor rates can be obtained, as shown below.

REMARK.

For simplicity's sake, the previous remarks do not consider

- interest rate swaps with the **EONIA** (Euro Overnight Index Average) as a floating rate;
- **repurchase agreements**, by which borrowers sell bonds to lenders with the obligation to repurchase them later at a usually higher price. The interest rate implicit in this contract is called repurchase rate and is usually slightly larger than the corresponding rate implicit in Treasury bonds, since this contract has a very small risk of default. In fact, should the lender (borrower) default, the borrower (lender) would withhold the loan (bonds). The repurchase rate of interest can vary in accordance with the bond deposited as security.

Estimation of implicit Euribor rates

Let time be measured in years, 0 be the time of estimation, $i_{0;T}$ and $\tilde{i}_{0;T}$ be a Euribor and a swap rate, set at time 0 for transactions with both term and maturity equal to *T*. Each known (unknown) Euribor $i_{0;T}$ with $T \leq 1$ (with $T > 1$) is a yearly rate of compound interest according to the **linear** (**exponential**) convention.

Some Euribor rates, implicit in the data of Table 7b, are estimated below; use is made of swap rates only, as in Hull (2012, chapt. 7). The estimation process comprises 3 stages

1) for each available term, a swap rate is obtained as an average of a bid and ask IRS rate. As a consequence, we have

$$\tilde{i}_{0;1} = 3{,}34\%;\ \tilde{i}_{0;2} = 3{,}15\%;\ \tilde{i}_{0;3} = 3{,}24\%;\ \tilde{i}_{0;4} = 3{,}36\%$$

2) for each missing term, equal to 1,5 / 2,5 / 3,5 years etc., a missing swap rate is obtained by linear interpolation, introduced below in Exercise **44**, point *b*. As a consequence, we have

$$\tilde{i}_{0;1,5} = \frac{\tilde{i}_{0;1} + \tilde{i}_{0;2}}{2} = 3{,}245\%; \quad \tilde{i}_{0;2,5} = \frac{\tilde{i}_{0;2} + \tilde{i}_{0;3}}{2} = 3{,}195\%; \quad \tilde{i}_{0;3,5} = \frac{\tilde{i}_{0;3} + \tilde{i}_{0;4}}{2} = 3{,}300\%$$

3) one unknown Euribor is obtained at a time by considering an interest rate swap agreed upon a time 0, with term equal to 1,5 years at first, 2 years then, and so forth. Keep in mind that both legs should have the same present value at trade time, which is calculated at the Euribor rates in our case. If we suppose that the notional capital is exchanged at the expiry of each swap, the floating (fixed) leg can be appraised, *mutatis mutandis*, as in Exercise **51**, point *a* (in Exercise **44**, point *c*), since it becomes like a floating (fixed) rate bond. As a consequence, we have

$$100 = 100\frac{\tilde{i}_{0;1,5}}{2}\left((1+i_{0;0,5}0{,}5)^{-1} + (1+i_{0;1})^{-1} + (1+i_{0;1,5})^{-1,5}\right) + 100(1+i_{0;1,5})^{-1,5} =$$
$$= 1{,}6225(1{,}01964^{-1} + 1{,}03978^{-1}) + 101{,}6225(1+i_{0;1,5})^{-1,5}$$

and hence $i_{0;1,5} = 3{,}260\%$ as well as

$$100 = 100\frac{\tilde{i}_{0;2}}{2}\left((1+i_{0;0,5}0{,}5)^{-1} + (1+i_{0;1})^{-1} + (1+i_{0;1,5})^{-1,5} + (1+i_{0;2})^{-2}\right) + 100(1+i_{0;2})^{-2} =$$
$$= 1{,}575(1{,}01964^{-1} + 1{,}03978^{-1} + 1{,}03260^{-1,5}) + 101{,}575(1+i_{0;2})^{-2}$$

and hence $i_{0;2} = 3{,}164\%$.

REMARK.

As explained in Hull (2012, chapt. 6), the intermediate section of a **money market** term structure, e.g. the one with terms falling between 3 and 15 months, may be estimated by taking out information from some futures on short term interest rates (e.g. the 3-month Euribor), as those derivative contracts are very liquid. Moreover, futures are free of credit risk, by virtue of the margin system managed by the clearinghouse of each futures exchange.

5.2. Forward rates of interest

Let time be measured in **years** and 0 be the present time. A **forward** rate $f_{0;t;T}$ **agreed upon today** is an interest rate **set today** that applies over an **agreed term** $[t;T]$ beginning at a **future time** *t*. It is charged at time *T* on a capital *C* lent over the term $[t;T]$ and repaid with a lump sum.

Forward rates are implicit in a term structure of **spot** interest rates $\{i_{0;t}\}$. For **arbitrage** to be ruled out, the same return must be yielded by all safe investment policies that are feasible today and have the same maturity *T*, with $0 < t < T$. Therefore, we have

$$1 + i_{0;T}T = (1 + i_{0;t}t)(1 + f_{0;t;T}(T - t)) \quad \text{with } \textbf{simple interest}$$

$$(1 + i_{0;T})^T = (1 + i_{0;t})^t (1 + f_{0;t;T})^{T-t} \quad \text{with } \textbf{compound interest}$$

$$i_{0;T}T = i_{0;t}t + f_{0;t;T}(T - t) \quad \text{with } \textbf{continuous compound interest}$$

As the present spot rates are known, the present forward rates can be obtained by solving the 3 **no arbitrage** equations. According to said equations, the following financial transactions, agreed upon at time 0, are equivalent: either lending a principal *C* for *T* years at the **spot** rate $i_{0;T}$ or first lending a principal *C* for *t* years at the **spot** rate $i_{0;t}$ and then its accumulation $C(1 + i_{0;t})$ for $T - t$ more years at the **forward** rate $f_{0;t;T}$. If it were not so, **arbitrage** would be achievable by borrowing a large amount on the less favourable terms and then lending it on the more favourable terms. As a consequence, no expenditure at time 0 would be matched by a considerable receipt at time *T*, equal to the difference between the accumulations of the 2 loans.
Forward rates can be agreed upon by entering into **interest rate derivatives** as borrowers or lenders of a (notional) principal over a future term.

Example 21.
Some yearly **spot** rates of **simple** interest on a particular day are

rate (%)	3,00	3,10	3,20	3,30
term	3 months	6 months	9 months	1 year

We want to find the 3x6, 6x9, and 6x12 **forward** rates, that can be agreed upon in a **forward rate agreement**; the term 3x6 starts (ends) 3 (6) months after trade. To reduce the credit risk of a **forward rate agreement**, no principal is exchanged and a cash settlement takes place at the beginning of the agreed term.

Solution.

Let time be measured in years and 0 be the present time. We have $i_{0;0,25} = 3\%;\ i_{0;0,5} = 3{,}1\%;\ i_{0;0,75} = 3{,}2\%;\ i_{0;1} = 3{,}3\%$. Substituting those values into the **no arbitrage** equations

$$1 + i_{0;0,5} 0{,}5 = (1 + i_{0;25} 0{,}25)(1 + f_{0;0,25;0,5} 0{,}25)$$

$$1 + i_{0;0,75} 0{,}75 = (1 + i_{0;5} 0{,}5)(1 + f_{0;0,5;0,75} 0{,}25)$$

$$1 + i_{0;1} = (1 + i_{0;5} 0{,}5)(1 + f_{0;0,5;1} 0{,}5)$$

and simplifying obtains $f_{0;0,25;0,5} = 3{,}176\%;\ f_{0;0,5;0,75} = 3{,}348\%;\ f_{0;0,5;1} = 3{,}447\%$.

According to the first **no arbitrage** equation, lending a principal *C* for 6 months at the **spot** rate $i_{0;0,5} = 3{,}1\%$ is the same as first lending a principal *C* for 3 months at the **spot** rate $i_{0;0,25} = 3\%$ and then its accumulation $C(1 + i_{0;0,25} 0{,}25)$ for 3 more months at the **forward** rate $f_{0;0,25;0,5} = 3{,}176\%$.

Although no allowance has been made for such frictions as taxes, bid-ask spreads, commissions, and fees, our **no arbitrage** equations provide a reasonable approximation for the **gross** accumulation of a **large** principal, as frictions are a small fraction of the total expense.

REMARK.

If the time pattern of the spot rates of interest were known, each forward rate of interest $f_{0;t;T}$, set at time 0, would be the same as the future spot rate of interest $i_{t;T}$, set at time *t* and charged over the term $[t;T]$. Under such conditions of **certainty**, the no arbitrage equations would define the notion of **consistency** for 2-variable accumulation factors. As proved in Section 1.3, a 2-variable accumulation factor is consistent, iff (if and only if) interest is **compounded**; for instance, we have

$$(1 + i_{0;3})^3 = (1 + i_{0;1})(1 + i_{1;2})(1 + i_{2;3}) = (1 + i_{0;1})(1 + i_{1;3})^2 = (1 + i_{0;2})^2 (1 + i_{2;3})$$

However, according to the empirical evidence each forward rate of interest $f_{0;t;T}$ would be equal to the sum of the expected value of the corresponding future spot rate of interest $i_{t;T}$ and a small liquidity premium; the longer the term $T-t$ of the loan, the larger said liquidity premium. As the time patterns of the spot rates of interest are not known with certainty, the no arbitrage condition can indeed be based on the rule of simple interest as well.

5.3. Appraisal of floating rate bonds

Let time be measured in **years** and *t* be the appraisal time with $0 \le t < 1$. Consider a **floating rate** bond with *n* remaining **yearly** coupons and face value of 100 **percent**; let the coupon rate be tied to the 1-year Euribor. As shown by the following diagram, the bond pays a coupon $100 i_{t-1;t}$ at the end of **year** *t* and the face value 100 at the **maturity** *n*; the coupon is unknown until time $t-1$, as $i_{t-1;t}$ is the 1-year Euribor quoted at time $t-1$ and charged on an unsecured interbank loan at time *t*.

		$100 i_{0;1}$	$100 i_{1;2}$		$100 i_{n-2;n-1}$	$100(1+i_{n-1;n})$
0	t	1	2	$\cdots$	$n-1$	n

Proposition.
The bond (clean) price is equal to 100 at issue and immediately after a coupon payment.

PROOF.
Let time be equal to $n-1$; the amount $100 i_{n-1;n} + 100$ is being paid after a year so that its present value at time $n-1$ and at the rate $i_{n-1;n}$ is 100 indeed. Let time be equal to $n-2$; the last-but-one coupon $100 i_{n-2;n-1}$ is accruing after a year, the bond price being 100, so that its present value at time $n-2$ and at the rate $i_{n-2;n-1}$ is 100 indeed. Repeating the process $n-2$ times attains time 0.

Let *t* be the present time; the **dirty price** of the floating rate bond is

$$P_{dirty} = P_{clean} + 100 i_{0;1} t = 100(1+i_{0;1})(1+i_{t;1}(1-t))^{-1}$$

as the term $1-t$ is not greater than a year and the Euribor $i_{t;1}$ is a rate of simple interest.

REMARK.

The above proposition applies to a **floating rate** bond that pays **half-yearly** (**quarterly**) coupons $m = 2$ $(m = 4)$ times per year, provided that the coupon rates are expressed on **half-yearly** (**quarterly**) basis. Let $0 \le t < \frac{1}{m}$; the above formula becomes

$$P_{dirty} = P_{clean} + 100 i_{0;1/m} t = 100\left(1 + \frac{i_{0;1/m}}{m}\right)\left(1 + i_{t;1/m}(\frac{1}{m} - t)\right)^{-1}$$

Exercise 44.

Some spot rates of interest for **interbank loans** on a particular day are

rate (%)	3,00	3,10	3,30	3,40	3,50	3,50	3,80
term	1 month	3 months	6 months	1 year	2 years	3 years	5 years

For simplicity's sake, each month has 30 days and there are no settlement lags.

a) Find the accumulation factor for an interbank deposit over a term of 6 months by assuming away the bid-ask spread.

b) Use **linear interpolation** to find an approximate discount factor for a sum due after 9 months.

c) Consider a 4% coupon bond issued by a AA-rated company, with yearly coupons and 3 years to maturity. Suppose that its quote is 100,70. Find its **credit spread**.

Solution.

Let time be measured in years and 0 be the present time. Let $i_{0;t}$ be the spot rate of interest for a transaction that starts at time 0 and ends at time *t*.

a) Since $i_{0;0,5} = 3{,}3\%$, a deposit of €1 over a term of 6 months brings about an accumulation of

$$1 + i_{0;0,5} 0{,}5 = 1{,}01650 \text{ €.}$$

b) To interpolate **linearly** the available spot rates, draw each pair (term; spot rate) in the above table as a point in the plane $(t; i_{0;t})$ and use line segments to join the points belonging to contiguous pairs. The resulting graph is piecewise linear. Since the 9 month maturity falls between the 6 month and 1 year maturities, the unknown spot rate $i_{0;0,75}$ is a function of $i_{0;0,5} = 0{,}033$ and $i_{0;1} = 0{,}034$. More precisely, it lies on the straight line

$$\frac{i_{0;t} - 0{,}033}{0{,}034 - 0{,}033} = \frac{t - 0{,}5}{1 - 0{,}5} \quad \text{namely} \quad i_{0;t} = 0{,}032 + 0{,}002t$$

that passes through the points $(0{,}5\,;0{,}033)$ and $(1\,;0{,}034)$ in the plane $(t\,;i_{0;t})$.

Since $i_{0;0,75} = 0{,}032 + 0{,}002 * 0{,}75 = 0{,}0335$ for $t = 0{,}75$, the required discount factor is

$$(1 + i_{0;0,75}0{,}75)^{-1} = (1 + 0{,}0335 * 0{,}75)^{-1} = 0{,}97549 .$$

c) The unknown spread *sp* satisfies the equation

$$4(1 + i_{0;1} + sp)^{-1} + 4(1 + i_{0;2} + sp)^{-2} + 104(1 + i_{0;3} + sp)^{-3} =$$
$$4(1{,}034 + sp)^{-1} + 4(1{,}035 + sp)^{-2} + 104(1{,}035 + sp)^{-3} = 100{,}70 \text{ €}$$

which has no closed-form solution. An approximate value of the **unique** credit spread is $sp = 0{,}25\%$.

Exercise 45.

Some spot rates of interest for **interbank loans** on a particular day are

rate (%)	3,00	3,10	3,30	3,40	3,50	3,50	3,80
term	1 month	3 months	6 months	1 year	2 years	3 years	5 years

For simplicity's sake, each month has 30 days and there are no settlement lags.

a) Find the equivalent nominal yearly rates of **continuous compound** interest for a term of either 6 months, 1 year or 2 years.

b) Use the nominal yearly rates of continuous compound interest and find the accumulation factor for a deposit over a term of 6 months by assuming away the bid-ask spread.

c) Use **exponential** interpolation to find an approximate discount factor for a sum due after 9 months.

Solution.

Let time be measured in years and 0 be the present time. Let $i_{0;t}$ be the spot rate of interest for a transaction that starts at time 0 and ends at time *t* under the assumption of continuous compounding.

a) We have

$$i_{0;0,5} = 2\ln\left(1+\frac{0,033}{2}\right) = 3,273\%;\ i_{0;1} = \ln(1+0,034) = 3,343\%;\ i_{0;2} = \ln(1+0,035) = 3,440\%$$

b) Since $i_{0;0,5} = 3,273\%$, a deposit of €1 over a term of 6 months brings about an accumulation of $e^{i_{0;0,5}*0,5} = 1,01650$ €. Owing to the equivalence of interest rates there is no difference with point **a**) of the previous exercise.

c) To interpolate **exponentially** the available spot rates, consider a table of nominal yearly rates of **continuous compound** interest, draw each pair (term; spot rate) in the table as a point in the plane $(t; i_{0;t})$ and use line segments to join the points belonging to contiguous pairs. The resulting graph is piecewise linear. Since the 9 month maturity falls between the 6 month and 1 year maturities, the unknown spot rate $i_{0;0,75}$ is a function of $i_{0;0,5} = 0,03273$ and $i_{0;1} = 0,03343$. More precisely, it lies on the straight line

$$\frac{i_{0;t} - 0,03273}{0,03343 - 0,03273} = \frac{t-0,5}{1-0,5} \quad \text{namely} \quad i_{0;t} = 0,03203 + 0,0014t$$

that passes through the points $(0,5\,; 0,03273)$ and $(1; 0,03343)$ in the plane $(t\,; i_{0;t})$.

Since $i_{0;0,75} = 0,03203 + 0,0014 * 0,75 = 0,03308$ for $t = 0,75$, the required discount factor is $e^{-i_{0;0,75}*0,75} = 0,97550$.

Exercise 46.

The prices of some zero-coupon bonds on a particular day are

price	98	96	94	92
term (**years**)	0,25	0,5	0,75	1

All bonds are Treasury bills; for simplicity's sake, each month has 30 days and there are no settlement lags. Find the spot rates of **compound** interest implicit in those prices.

Solution.

Let time be measured in years and 0 be the present time. Let $i_{0;t}$ be the spot rate of interest for a transaction that starts at time 0 and ends at time *t*.

From the equation $\text{price} = 100(1+i_{0;t})^{-t}$ we get

$$1+i_{0;t} = \left(\frac{100}{\text{price}}\right)^{\frac{1}{t}}$$

where $\frac{100}{\text{price}} - 1$ is a periodic interest rate.

Substituting the above prices in the latter equation obtains the following table

periodic rate	2,04%	4,17%	6,38%	8,70%
effective rate $i_{0;t}$	8,42%	8,51%	8,60%	8,70%
term (**years**)	0,25	0,5	0,75	1

Exercise 47.

The prices of some zero-coupon bonds on a particular day are

price	99	98	97	96
term (**years**)	0,75	1,25	1,75	2,25

All bonds are Treasury bonds; for simplicity's sake, each month has 30 days and there are no settlement lags. Find

a) the spot rates of **compound** interest implicit in those prices;

b) the 1-year spot rate of **compound** interest by means of linear interpolation.

Solution.

Let time be measured in years and 0 be the present time. Let $i_{0;t}$ be the spot rate of interest for a transaction that starts at time 0 and ends at time *t*.

a) From the equation $\text{price} = 100(1+i_{0;t})^{-t}$ we get

$$1+i_{0;t} = \left(\frac{100}{\text{price}}\right)^{\frac{1}{t}}$$

where $\frac{100}{\text{price}} - 1$ is a periodic interest rate.

Substituting the above prices in the latter equation obtains the following table

periodic rate	1,01%	2,04%	3,09%	4,17%
effective rate $i_{0;t}$	1,35%	1,63%	1,76%	1,83%
term (**years**)	0,75	1,25	1,75	2,25

b) We have

$$i_{0;1} = i_{0;0,75} + \left(i_{0;1,25} - i_{0;0,75}\right)\frac{0,25}{0,5} = 1,35\% + 0,28\%\frac{1}{2} = 1,49\% .$$

REMARK.

Financial intermediaries may strip *n* half-yearly coupons from the face value of a Treasury bond, thus generating $n+1$ individual zero coupon bonds. Such a transaction may be registered with a **central electronic custodian**. Such a **coupon stripping** was regulated by the US Treasury in 1985 and the Italian Treasury in 1998.

REMARK.

When it comes to an economic analysis, optimisation is generally used to estimate the unknown parameters of a function of the term *t*, either a **parsimonious** or a piecewise polynomial (exponential) one; the latter is called **spline**.

A well known instance of a **parsimonious** function is put forward in Nelson-Siegel (1987); the resulting term structure can display a monotonic, humped, or sigmoidal shape

$$i_{0;t} = a \ + \ b\frac{t}{\tau}\left(1 - \exp\left(-\frac{t}{\tau}\right)\right) \ + \ c\exp\left(-\frac{t}{\tau}\right)$$

where $i_{0;t}$ is a nominal rate of continuous compound interest and *a*, *b*, *c*, and τ are the unknown parameters.

As for a **spline**, the various pieces of a term structure and hence the short and long-term spot rates $i_{0;t}$ are somewhat independent. Owing to risk management needs, practioners often take a few pieces into account, the representative maturities being 1, 3, 5, 7, 10, and 30 years (see Marangio et al., 2002).

Exercise 48.

Suppose that some yearly **spot** rates on a particular day are

rate (%)	5,00	5,10	5,30	5,40	5,50	5,50	5,80
term	1 month	3 months	6 months	1 year	2 years	3 years	5 years

Calculate the yearly **forward** rate 3x12 under the assumption that the above data represent

a) yearly rates of compound interest according to the **linear** convention;

b) yearly rates of compound interest according to the **exponential** convention;

c) nominal yearly rates of continuous compound interest.

Solution.

Let time *t* be measured in years and 0 be the present time. Let $i_{0;t}$ be the **spot** rate of interest for a transaction that starts at time 0 and ends at time *t*; let $f_{0;t;T}$ be the **forward** rate of interest set at time 0 for a transaction that starts at time *t* and ends at time *T*. The term 3x12 starts (ends) 3 (12) months after trade. Substituting the values $i_{0;0,25} = 5{,}1\%$; $i_{0;1} = 5{,}4\%$ into the **no arbitrage** equation

a) $\left(1 + i_{0;0,25} 0{,}25\right)\left(1 + f_{0;0,25;1} 0{,}75\right) = 1 + i_{0;1}$

b) $\left(1 + i_{0;0,25}\right)^{0,25}\left(1 + f_{0;0,25;1}\right)^{0,75} = 1 + i_{0;1}$

c) $e^{i_{0;0,25} 0,25} e^{f_{0;0,25;1} 0,75} = e^{i_{0;1}}$

and simplifying obtains

a) $f_{0;0,25;1} = \dfrac{1}{0{,}75}\left(\dfrac{1 + i_{0;1}}{1 + i_{0;0,25} 0{,}25} - 1\right) = 5{,}431\%$

b) $f_{0;0,25;1} = \left(\dfrac{1 + i_{0;1}}{\left(1 + i_{0;0,25}\right)^{0,25}}\right)^{\frac{1}{0,75}} - 1 = 5{,}500\%$

c) $f_{0;0,25;1} = \dfrac{i_{0;1} 1 - i_{0;0,25} 0{,}25}{0{,}75} = 5{,}500\%$

Recall that the **forward** rate of **simple** interest $f_{0;0,25;1} = 5{,}431\%$ can be agreed upon at time 0 by the 2 counterparts in a **forward rate agreement**.

Exercise 49.

Six months ago a 6x12 FRA (forward rate agreement) was negotiated on a notional principal of €100.000. Some yearly spot rates of **simple interest** are reported in the table below. For simplicity's sake, each month has 30 days and there are no settlement lags. Describe the cash settlement that takes place today.

term	6 months	1 year
rate (6 months ago)	4,00%	5,00%
rate (today)	4,50%	5,50%

Solution.

If the principal were exchanged, this would happen twice, 6 and 12 months after negotiation (which explains the expression 6x12). However, all of this is replaced by a cash settlement 6 months after negotiation.

Let time be measured in years, 0 be the negotiation time and 0,5 be the settlement time (that falls today). Let $i_{0;t}$ be the **spot** rate of **simple** interest for a transaction that starts at time 0 and ends at time *t*; let $f_{0;t;T}$ be the **forward** rate of **simple** interest set at time 0 for a transaction that starts at time *t* and ends at time *T*. Substituting the values $i_{0;0,5} = 4\%$; $i_{0;1} = 5\%$ into the no arbitrage equation obtains

$$1 + f_{0;0,5;1} 0,5 = \frac{1 + i_{0;1}}{1 + i_{0;0,5} 0,5} = \frac{1,05}{1,02}$$

from which it follows that the agreed forward rate is $f_{0;0,5;1} = 5,882\%$. Since $i_{0,5;1} = 4,50\%$, the borrower must pay

$$\frac{100.000\left(f_{0;0,5;1} - i_{0,5;1}\right)0,5}{1 + i_{0,5;1} 0,5} = \frac{100.000(5,882\% - 4,500\%)0,5}{1 + 4,500\% * 0,5} = 675,79 \text{ €}$$

to the lender. However, if €100.675,79 are then borrowed over a 3 month term at the spot rate $i_{0,5;1}$, the repayment at maturity will amount to

$$100.675,79(1 + 4,5\% * 0,5) = 100.000(1 + 5,882\% * 0,5) = 102.941,00 \text{ €}$$

as implicit in the FRA. If $i_{0,5;1}$ had been greater than 5,882%, the lender should have paid the amount $\frac{100.000\left(i_{0,5;1}-f_{0;0,5;1}\right)0,5}{\left(1+i_{0,5;1}0,5\right)}$ to the borrower.

Exercise 50.

In order to repay a loan of €96.000 1 year from now, a company could deposit with a bank a receipt of €40.000 6 months from now as well as a receipt of €55.000 9 months from now. Today's spot rates of **simple** interest per year are

rate (%)	4,00	4,10	4,20	4,30
term	3 months	6 months	9 months	1 year

The interest rate risk would be hedged by negotiating a 6x12 and a 9x12 forward rate agreement. Find whether the financial transactions are effective.

Solution.

The financial transactions are effective. Let time be measured in years and 0 be the present time. Let $i_{0;t}$ be the **spot** rate of **simple** interest for transactions that start now and end at time *t*; we have $i_{0;0,5}=0,041;\ i_{0;0,75}=0,042;\ i_{0;1}=0,043$. Recall that an ideal **arbitrage** is a set of simultaneuos financial transactions that does not require any (down)payments and does or might provide some receipt. For **arbitrage** to be impossible in our case, any set of simultaneuos safe transactions agreed upon today and concerning the above-mentioned receipts has to bring forth the same accumulation 12 months from now. As a consequence, the negotiation of a 6x12 and a 9x12 forward rate agreements amounts also to discounting both receipts to time 0 and then depositing the resulting present value until time 1

$$\left(40.000\left(1+i_{0;0,5}0,5\right)^{-1}+55.000\left(1+i_{0;0,75}0,75\right)^{-1}\right)\left(1+i_{0;1}\right)=96.495,11>96.000$$

REMARK.

This is the simplest possible procedure for calculating the safe accumulation of future and sure receipts based on the spot rates quoted on a particular day.

Exercise 51.

Consider a **floating rate** bond with credit rating AA, face value of 100 percent, yearly coupons, and 15 months to maturity. Some spot rates of interest for **interbank loans** quoted 9 months ago and today are reported in the table below

term	3m	6m	1y	2y	3y	5y
rate (**9 months ago**, %)	2,60	2,80	3,00	3,10	3,10	3,15
rate (**today**, %)	2,70	2,90	3,10	3,20	3,20	3,25

Find the clean price of the bond under the assumption that the coupon rate is equal to a 1-year spot rate

a) with no spread;

b) with a spread of 20 basis points (=0,20 %).

Solution.

Let time be measured in years, 0,75 be the present time and 2 be the bond expiry. The latest coupon was paid 9 months ago at time 0. Let $i_{0;1} = 3{,}000\%$ be the 1-year spot rate quoted 9 months ago and $i_{0,75;1} = 2{,}700\%$ be the 3-month spot rate quoted now.

a) The next coupon, worth $100 i_{0;1} = 3$, will accrue 3 months from now at time 1; the clean price of the bond will then be 100. The dirty and clean prices of the bond are

$$100(1 + i_{0;1})(1 + i_{0,75;1} 0{,}25)^{-1} = 103(1 + 0{,}027 * 0{,}25)^{-1} = 102{,}31$$

and

$$102{,}31 - 2{,}25 = 100{,}06$$

with $100 i_{0;1} 0{,}75 = 2{,}25$ being accrued interest.

b) Let $i_{0,75;1} = 2{,}700\%$; $i_{0;75;2} = 3{,}1\% + 0{,}1\% \frac{0{,}25}{1} = 3{,}125\%$ be the 3-month and 15-month spot rates quoted now, with the latter being obtained by linear interpolation. This floating rate bond includes two components: the floating rate bond of point **a**) as well as a sequence of 2 payments, each worth $100 * 0{,}0020 = 0{,}20$, due 3 and 15 months from now. The dirty and clean prices of the bond are therefore

$$102{,}31 + 0{,}20\left((1 + i_{0,75;1} 0{,}25)^{-1} + (1 + i_{0,75;2})^{-1{,}25}\right) = 102{,}70$$

and

$$102{,}70 - 2{,}40 = 100{,}30$$

with $100(i_{0;1} + 0{,}0020)0{,}75 = 2{,}40$ being accrued interest.

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